The *Approximate Irreducible Factorization*

of a Univariate Polynomial. Revisited

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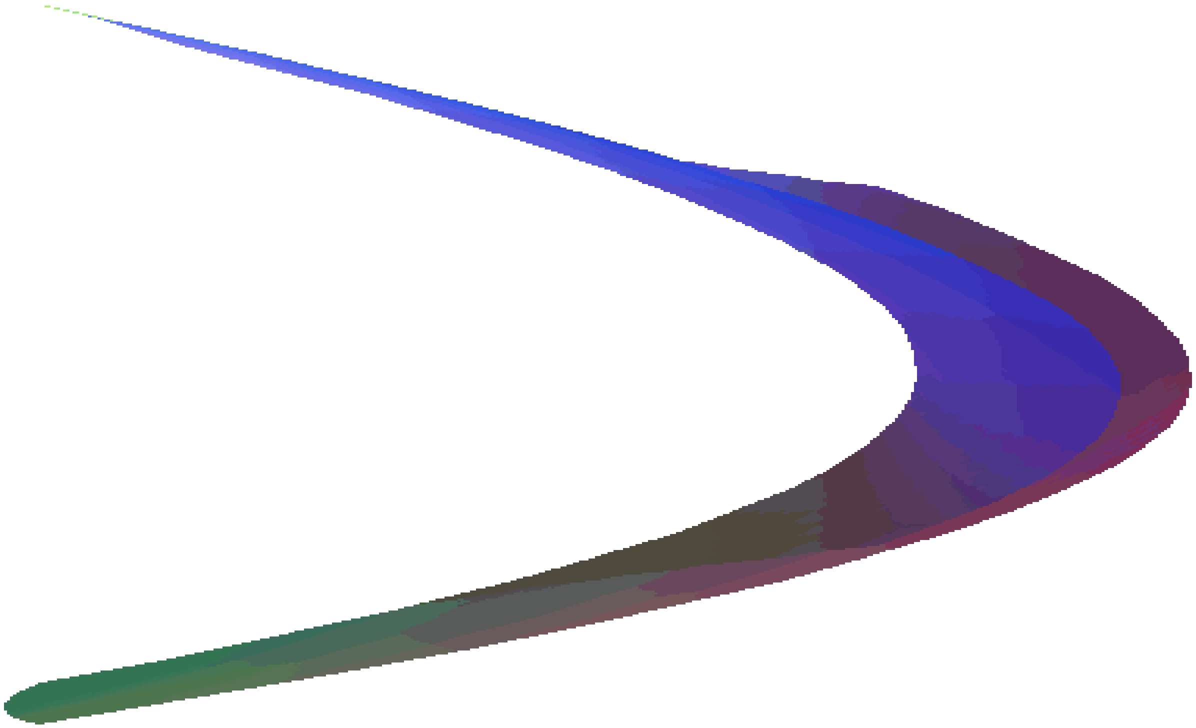
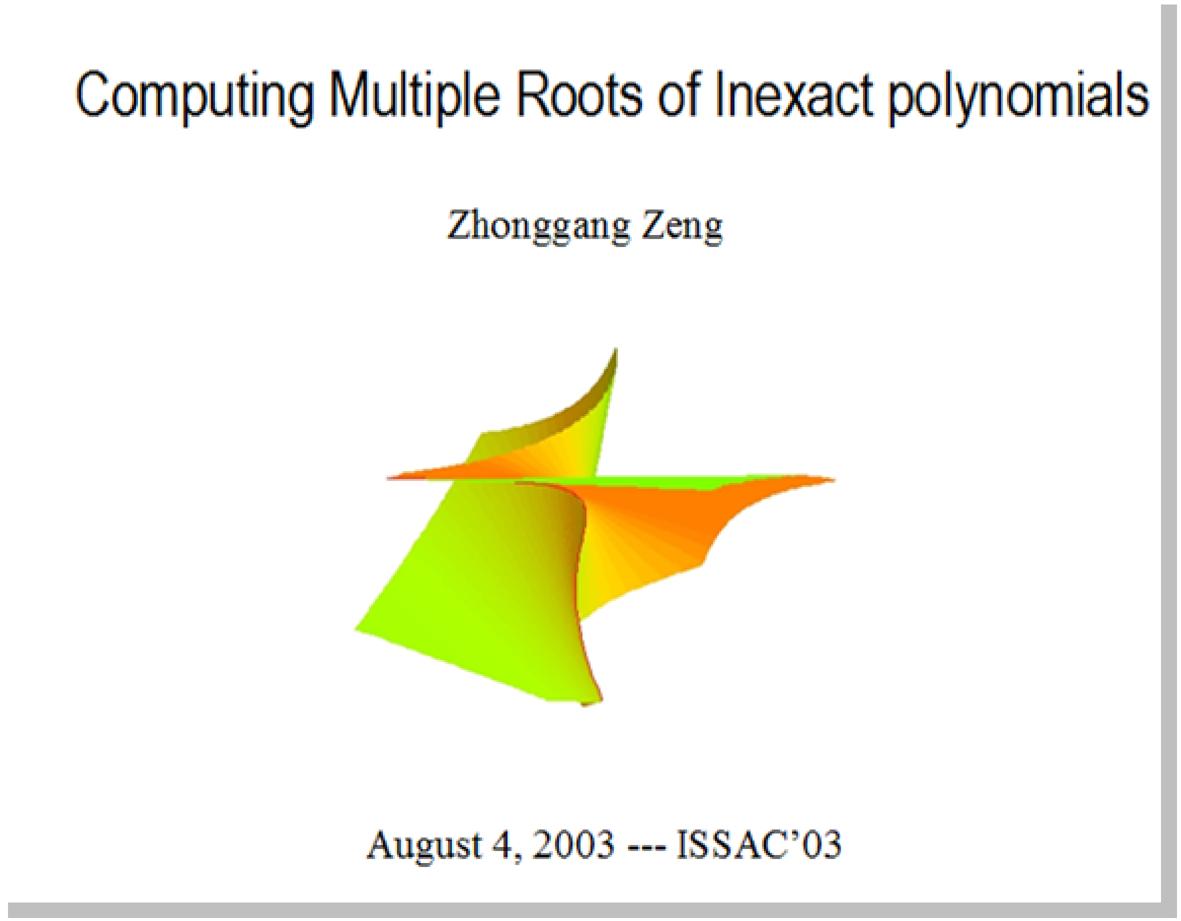


ISSAC 2009, KIAS, Seoul, July 31, 2009

(supported in part by NSF under Grant DMS-0715137)

This talk revisits the work



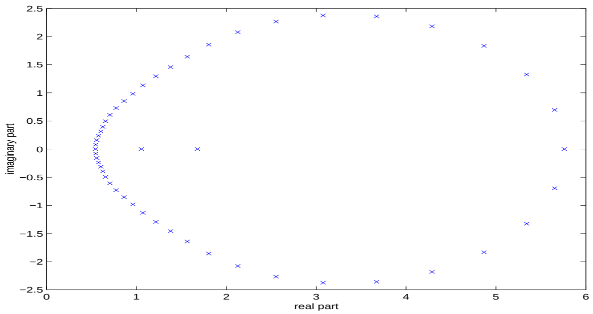
An algorithm for accurate computation of multiple roots and multiplicities using floating point arithmetic even if the coefficients are perturbed.

A two-staged algorithm proposed in ISSAC 2003

For the polynomial

$$(x-1)^{20}(x-2)^{15}(x-3)^{10}(x-4)^{5}$$

with (inexact) coefficients in machine precision



Stage I results:

The backward error: 6.05×10^{-10}

Computed roots	multiplicities
1.000000000000353	20
2.000000000030904	15
3.000000000176196	10
4.000000000109542	5

Stage II results:

The backward error: 6.16×10^{-16}

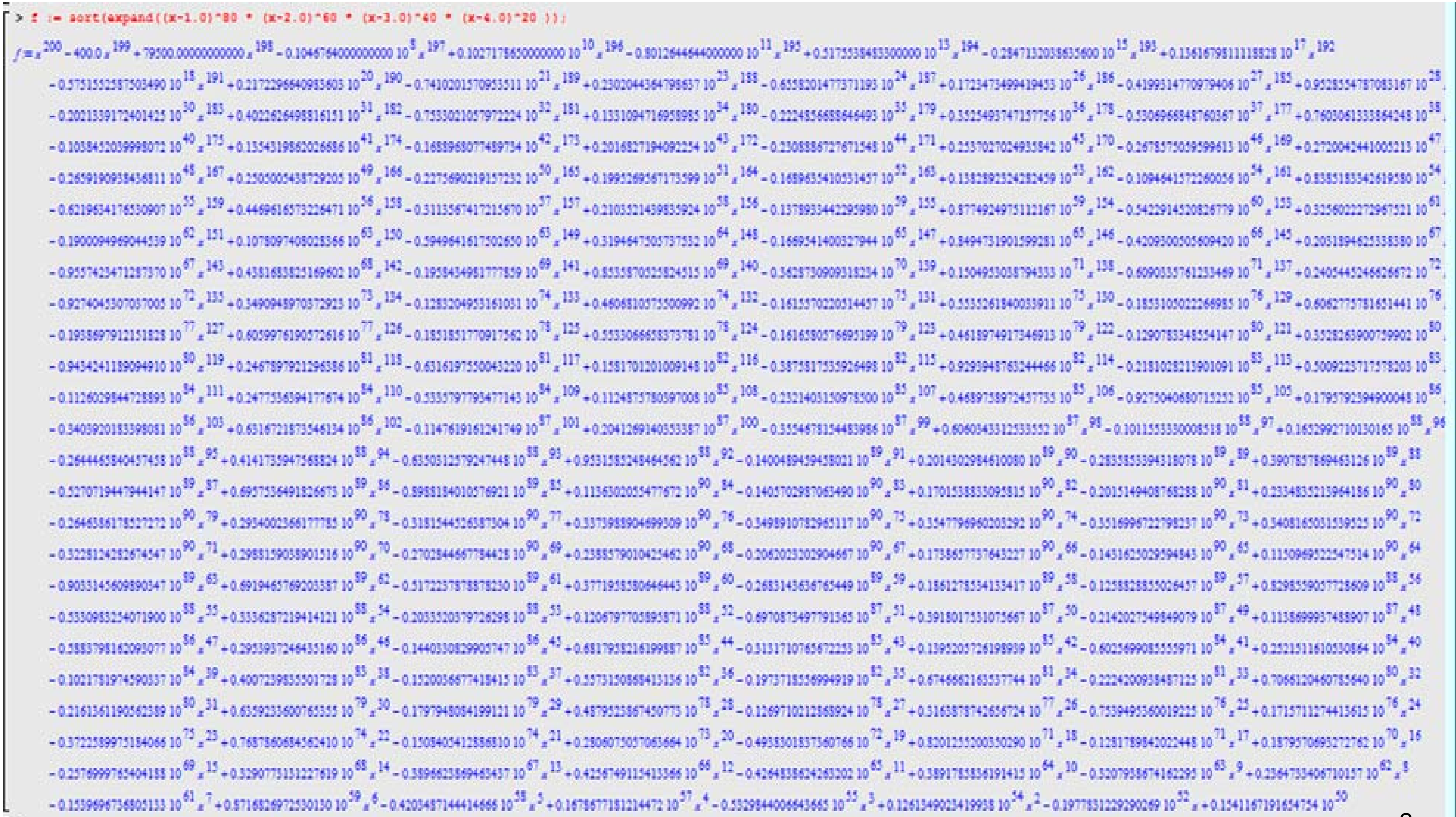
Computed roots	multiplicities
1.000000000000000	20
1.999999999999997	15
3.000000000000011	10
3.999999999999985	5

Software package MultRoot: Z. Zeng, ACM TOMS 2004

Example of a new method: For polynomial

$$(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$$

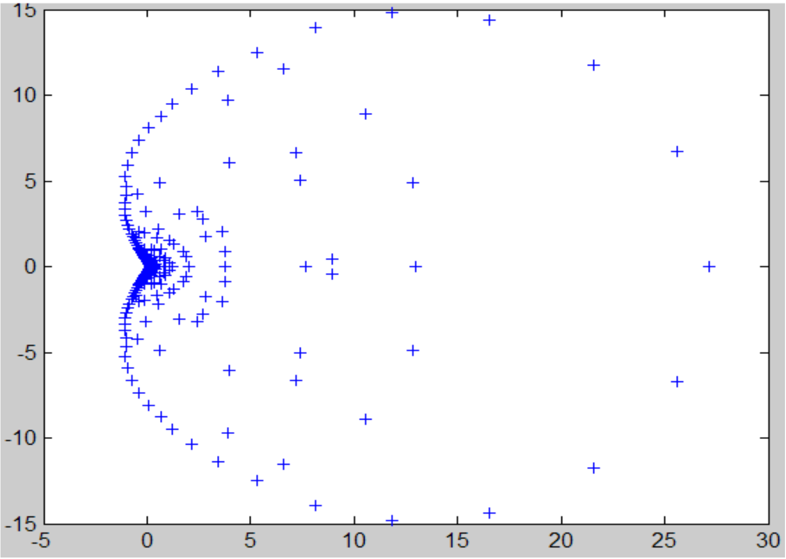
with (inexact) coefficients in hardware precision



Example: For polynomial $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

with (inexact) coefficients in hardware precision

Exact factorization



Approximate factorization:

```
>> [F,res,fcnd]  =  uvFactor(f,1e-10,1);

THE CONDITION NUMBER:                          914.329
THE BACKWARD ERROR:                     5.71e-015
THE ESTIMATED FORWARD ROOT ERROR:       1.04e-011

  FACTORS

   ( x -     3.999999999999990 )^20
   ( x -     3.000000000000008 )^40
   ( x -     1.999999999999998 )^60
   ( x -     1.000000000000000 )^80
```

A significant advancement in robustness
but *not really the point* for a revisit

Question: What problem are we really solving?

The Approximate Irreducible Factorization (also known as Root-finding) Problem:

Given

$$p(x) = x^6 + .6667\,x^5 - 2.333\,x^4 - 1.333\,x^3 + 1.667\,x^2 + 0.6667\,x - 0.3333$$

Conventional Factorization

$$(x+1.0189+0.0034i)(x+1.019-0.00336i)(x+0.9621)(x-0.3332)(x-1+0.0144i)(x-1-0.0144i)$$

Small perturbation

$$\hat{p}(x) = (x-1)^2(x+1)^3\left(x-\tfrac{1}{3}\right)^1$$

$$= \hat{a}_0(\hat{a}_1 x+\hat{b}_1)^2(\hat{a}_2 x+\hat{b}_2)^3(\hat{a}_3 x+\hat{b}_3)^1$$

AIF

well-posed?

1. Match multiplicities

2. $\left|\hat{a}_i-\tilde{a}_i\right|+\left|\hat{b}_i-\tilde{b}_i\right| = O\left(\left\|p-\tilde{p}\right\|\right)$

$$\tilde{p}(x) = \tilde{a}_0(\tilde{a}_1 x+\tilde{b}_1)^2(\tilde{a}_2 x+\tilde{b}_2)^3(\tilde{a}_3 x+\tilde{b}_3)^1$$

$$= 0.9999(1.0x-1.00001)^2(1.0x+1.000009)^3(1.0x-0.3334)^1$$

A well-posed problem: (Hadamard, 1923)

the solution satisfies

- existence
- uniqueness
- continuity w.r.t data

An ill-posed problem is *infinitely* sensitive to perturbation

tiny perturbation → huge error

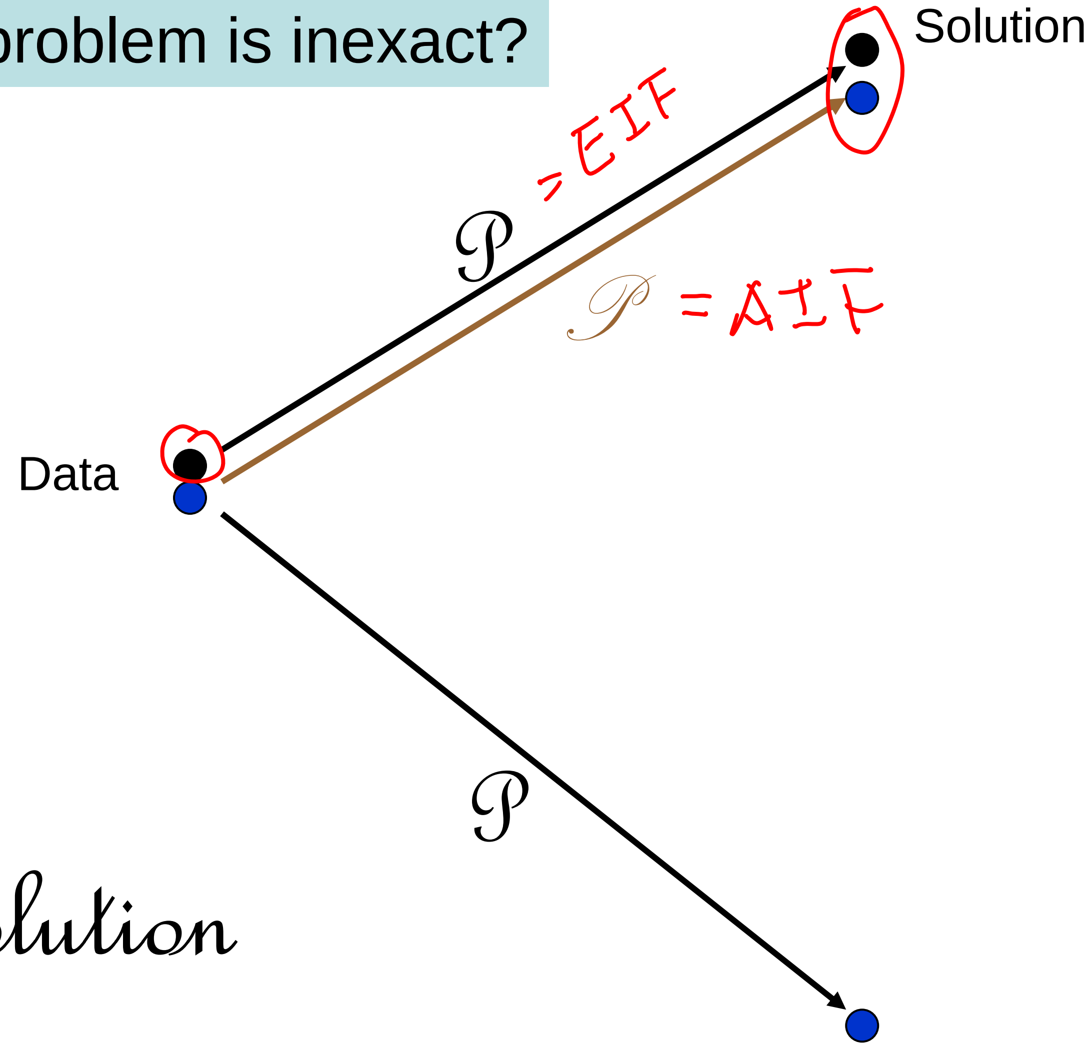
A frontier in scientific computing

Though frequently needed in application, the adequate handling of such ill-posed … problems is hardly ever touched upon in numerical analysis textbooks.

--- Arnold Neumaier, SIAM Review

Challenge in solving ill-posed problems:

Can we recover the lost solution when the problem is inexact?

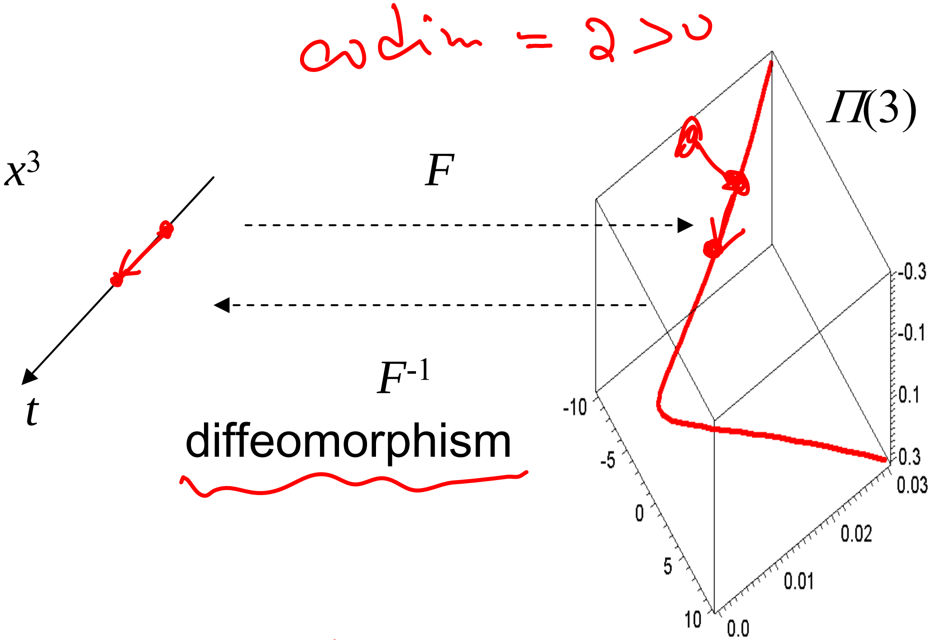


$$\mathcal{P} : \mathfrak{Data} \rightarrow \mathfrak{Solution}$$

Geometry of AIF (simplified view)

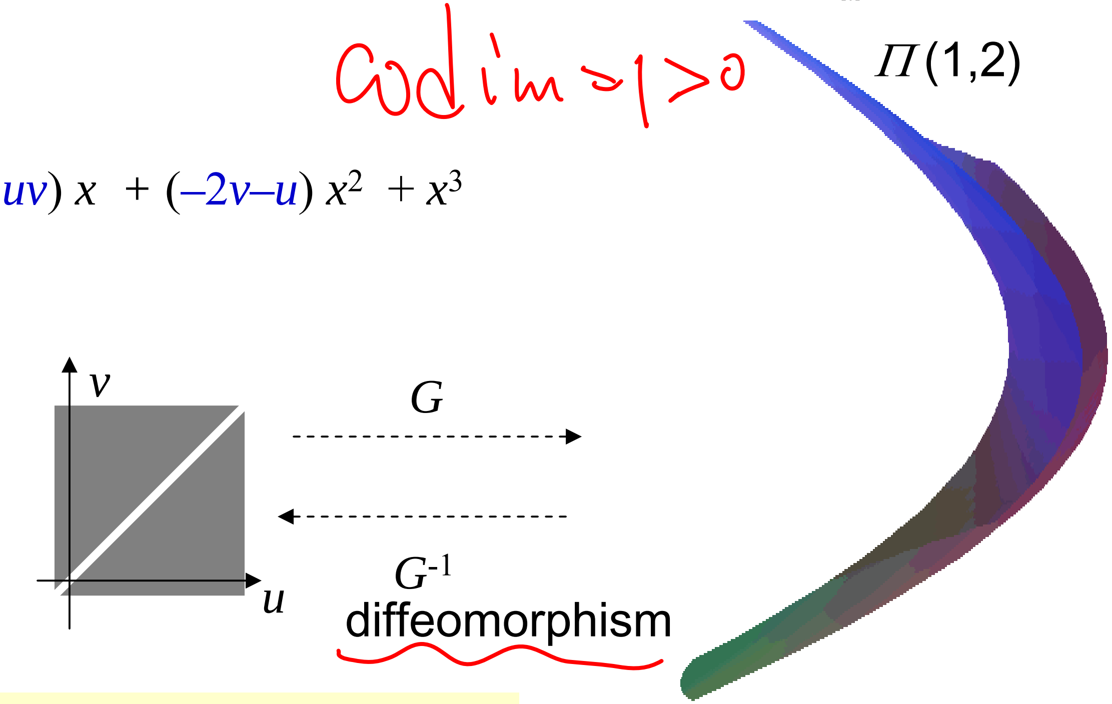
$(x-t)^3 = -t^3 + (3t^2)\,x + (-3t)\,x^2 + x^3$

$$F(t) = \begin{bmatrix} -t^3 \\ 3t^2 \\ -3t \end{bmatrix}$$



$(x-u)^1\,(x-v)^2 = -uv^2 + (v^2 + 2uv)\,x + (-2v-u)\,x^2 + x^3$

$$G(u,v) = \begin{bmatrix} -uv^2 \\ v^2 + 2uv \\ -2v-u \end{bmatrix}$$



Polynomials form (factorization) manifolds

Proposition 1: Polynomials

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m$$

$$= a_0 (a_1 x + \hat{b}_1)^{k_1} (a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n}$$

form a differentiable manifold $\Pi_{[k_1, \ldots, k_n]}$ of codimension

codim $(\Pi_{[k_1, \ldots, k_n]})$ = m - n

dimension of the polynomial vector space (≥ polynomial degree)

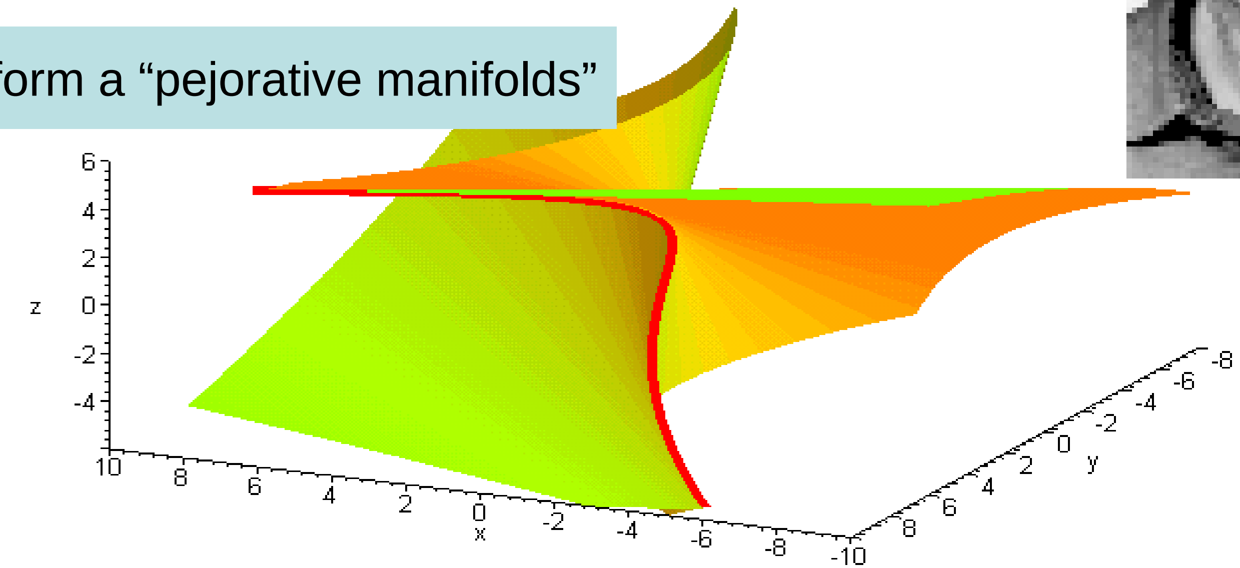
Number of factors

Are ill-posed problems really sensitive?

Kahan: It is a misconception.

W. Kahan’s observation (1972)

- Problems form a “pejorative manifolds”



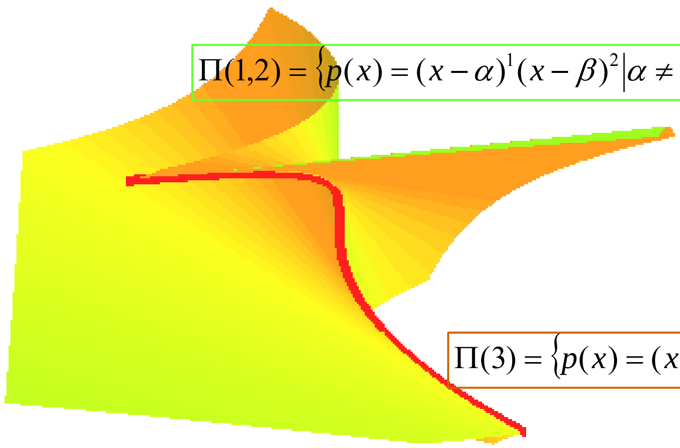
Plot of pejorative manifolds of degree 3 polynomials with multiple roots

- Ill-posedness: a tiny perturbation pushes the problem out of the manifold

- A problem is <u>not</u> sensitive at all if it stays on the manifold.

Stratification of factorization manifolds of degree 3 monic polynomials

$\Pi(1,1,1)=\{p(x)=(x-\alpha)^1(x-\beta)^1(x-\gamma)^1 | \alpha\neq\beta\neq\gamma\}$



$$\overline{\Pi(3)} \subset \overline{\Pi(1,2)} \subset \overline{\Pi(1,1,1)} = C^3$$

Codimensions: 2 1 0

Factorization manifold stratification of degree 4 polynomials:

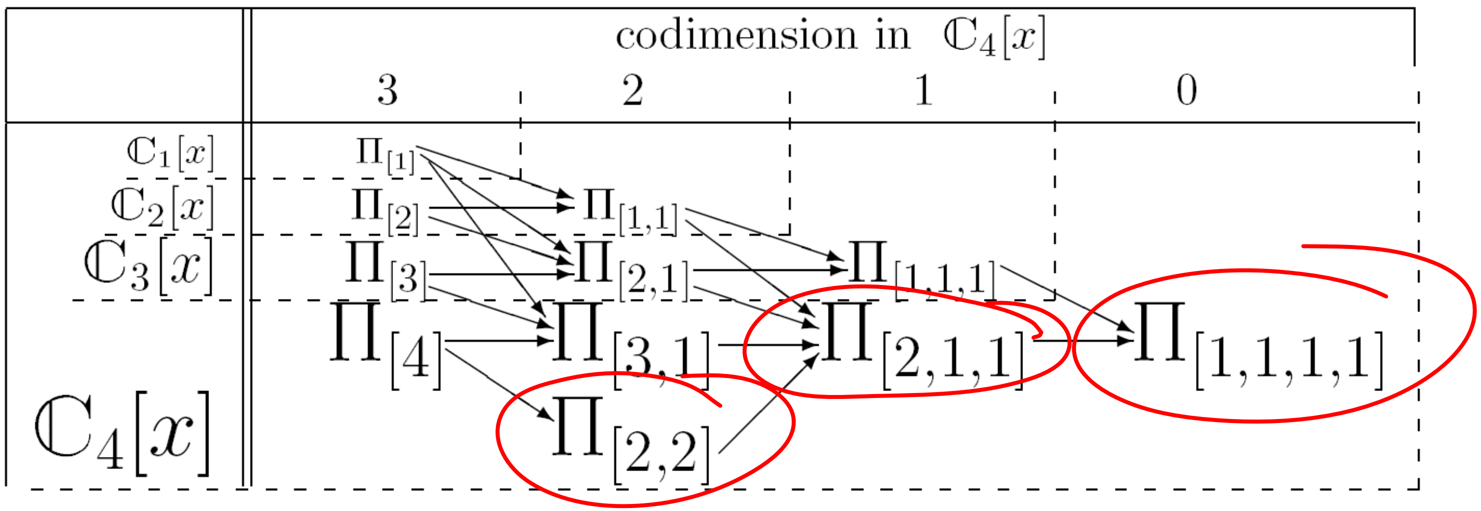
$$\overline{\Pi(4)} \subset \begin{matrix}\overline{\Pi(2,2)} \\ \overline{\Pi(1,3)}\end{matrix} \subset \overline{\Pi(1,1,2)} \subset \overline{\Pi(1,1,1,1)} = C^4$$

Codimensions: 3 2 1 0

Factorization manifolds and their stratification

$$\Pi_{[k_1k_2\cdots k_n]} = \left\{ a_0(a_1x+b_1)^{k_1}(a_2x+b_2)^{k_2}\cdots(a_nx+b_n)^{k_n} \;\middle|\; a_i,b_i \in C, a_ib_j \neq a_jb_i, \forall i \neq j \right\}$$

$$\subset \quad C_m[x] = \left\{ c_0 + c_1x + \cdots + c_mx^m \;\middle|\; c_i \in C \right\}$$



$$p \in \Pi_{[2,2]} \quad \rightleftarrows \quad \mathrm{dist}(p, \Pi_{[2,2]}) = \mathrm{dist}(p, \Pi_{[2,1,1]}) = \mathrm{dist}(p, \Pi_{[1,1,1,1]}) = 0$$

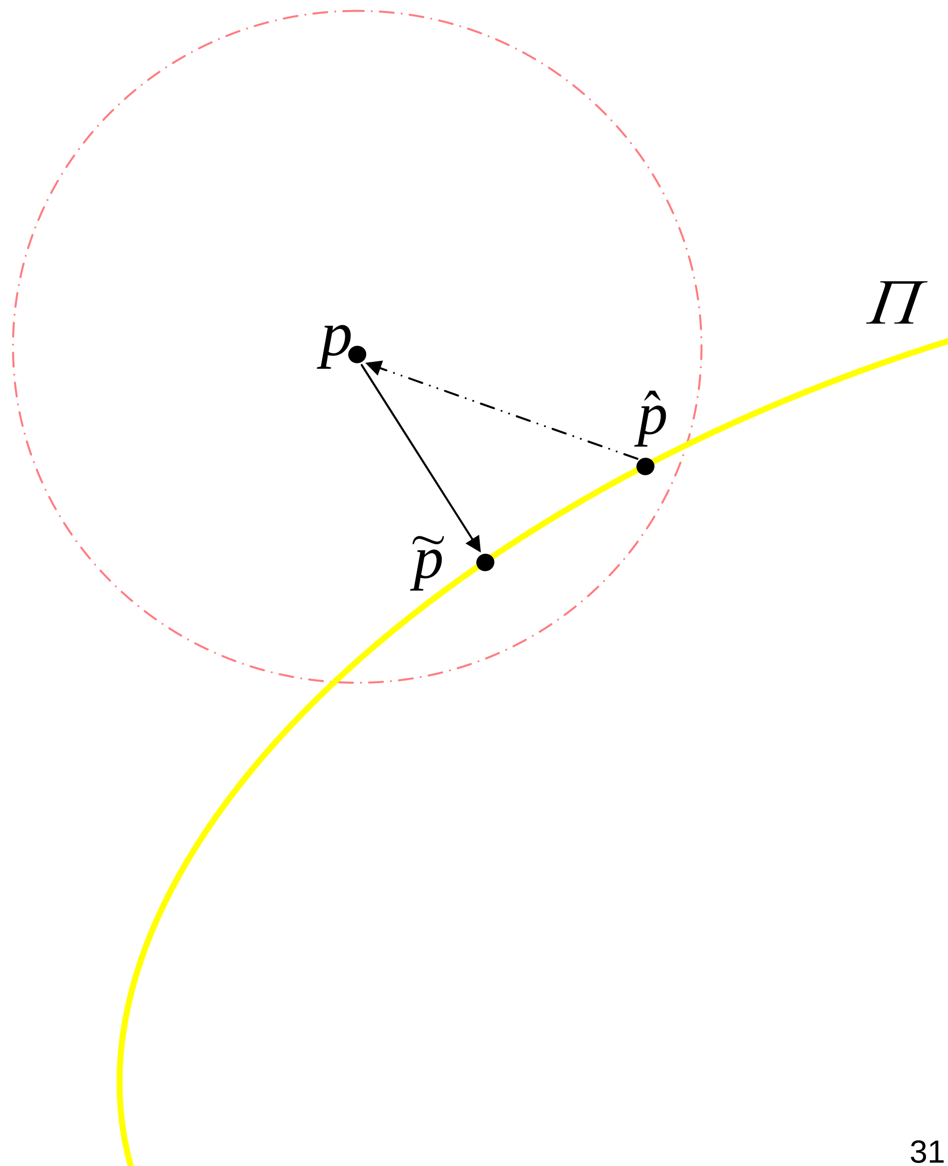
Proposition 3: $p \in \Pi_{[k_1 \ldots k_n]}$ if and only if

$$\mathrm{codim}(\Pi_{[k_1 \ldots k_n]}) = \max \{ \mathrm{codim}(\Pi) \mid \mathrm{dist}(p, \Pi) = 0 \}$$

Formulation of the approximate irreducible factorization

The approximate factorization of p is

- the exact factorization of $\tilde{p}$
- $\tilde{p}$ lies in the nearby manifold $\prod$ of the highest codimension
- $\tilde{p}$ is the nearest polynomial on $\prod$ from p



A "three-strikes" principle for formulating
an approximate irreducible factorization:

- **Backward nearness**: The AIF is the exact factorization of a nearby polynomial
- **Maximum codimension**: The AIF is the exact factorization of a polynomial in the nearby factorization manifold of the highest codimension.
- **Minimum distance**: The AIF is the exact factorization of the nearest polynomial in the nearby factorization manifold of the highest codimension.

In comparison:

Symbolic computation:

Backward nearness with distance = 0

Numerical computation: (straightforward)

Backward nearness with minimal distance

- Finding the AIF is (apparently) a well-posed problem
- The AIF is a generalization of exact factorization.

Theorem 1

$$\hat{p}(x) = \hat{a}_0 (\hat{a}_1 x + \hat{b}_1)^{k_1} \cdots (\hat{a}_n x + \hat{b}_n)^{k_n} \in \Pi_k$$

Assume $\|p - \hat{p}\|$ is small

Then, $\exists$ an interval I and $\forall \varepsilon \in I$

$\exists$ a unique AIF within ε

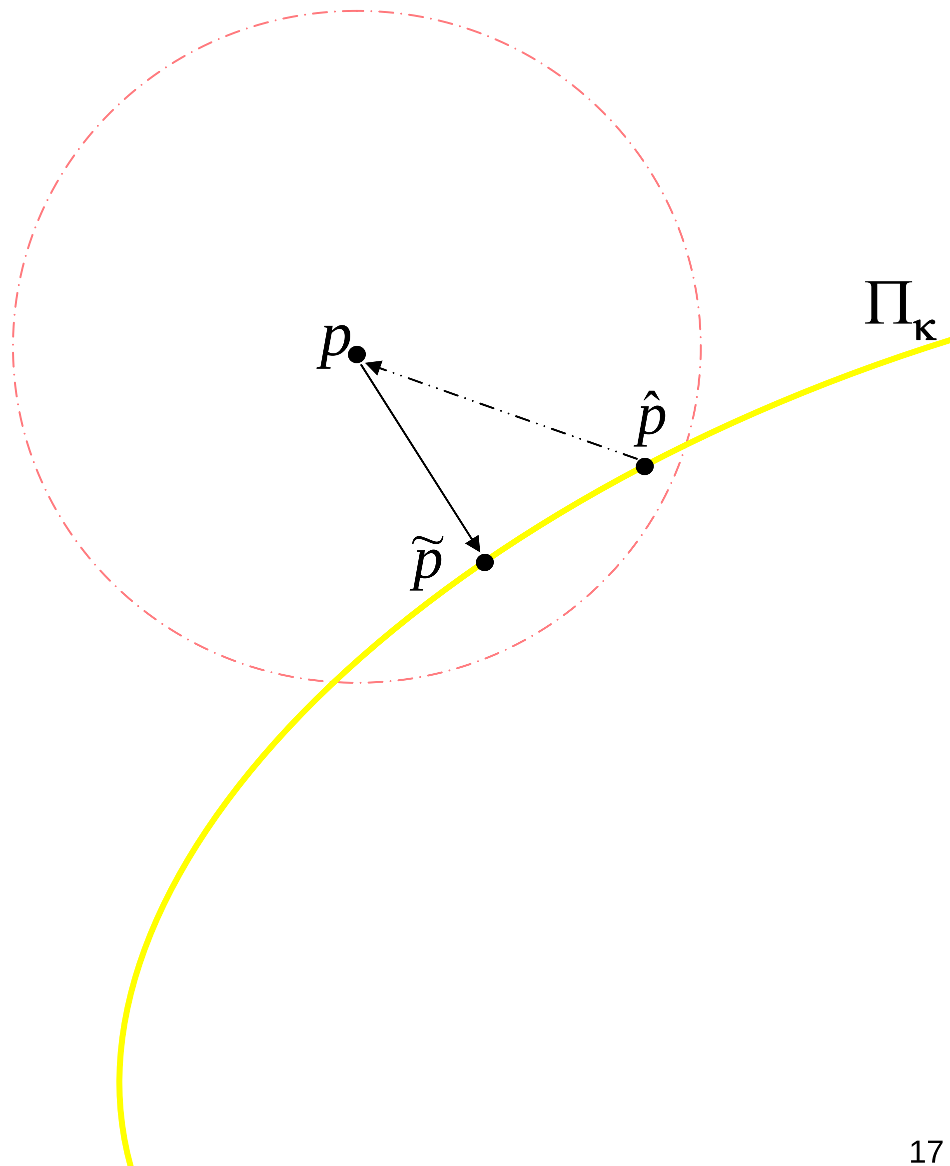
$$\tilde{p}(x) = \tilde{a}_0 (\tilde{a}_1 x + \tilde{b}_1)^{k_1} \cdots (\tilde{a}_n x + \tilde{b}_n)^{k_n} \in \Pi_k$$

such that

$$\left\| (\tilde{a}_i x + \tilde{b}_i) - (\hat{a}_i x + \hat{b}_i) \right\| = O\left(\|p - \hat{p}\| \right)$$

Moreover, the AIF is continuous w.r.t. p

Liptchtz?



The two-staged algorithm

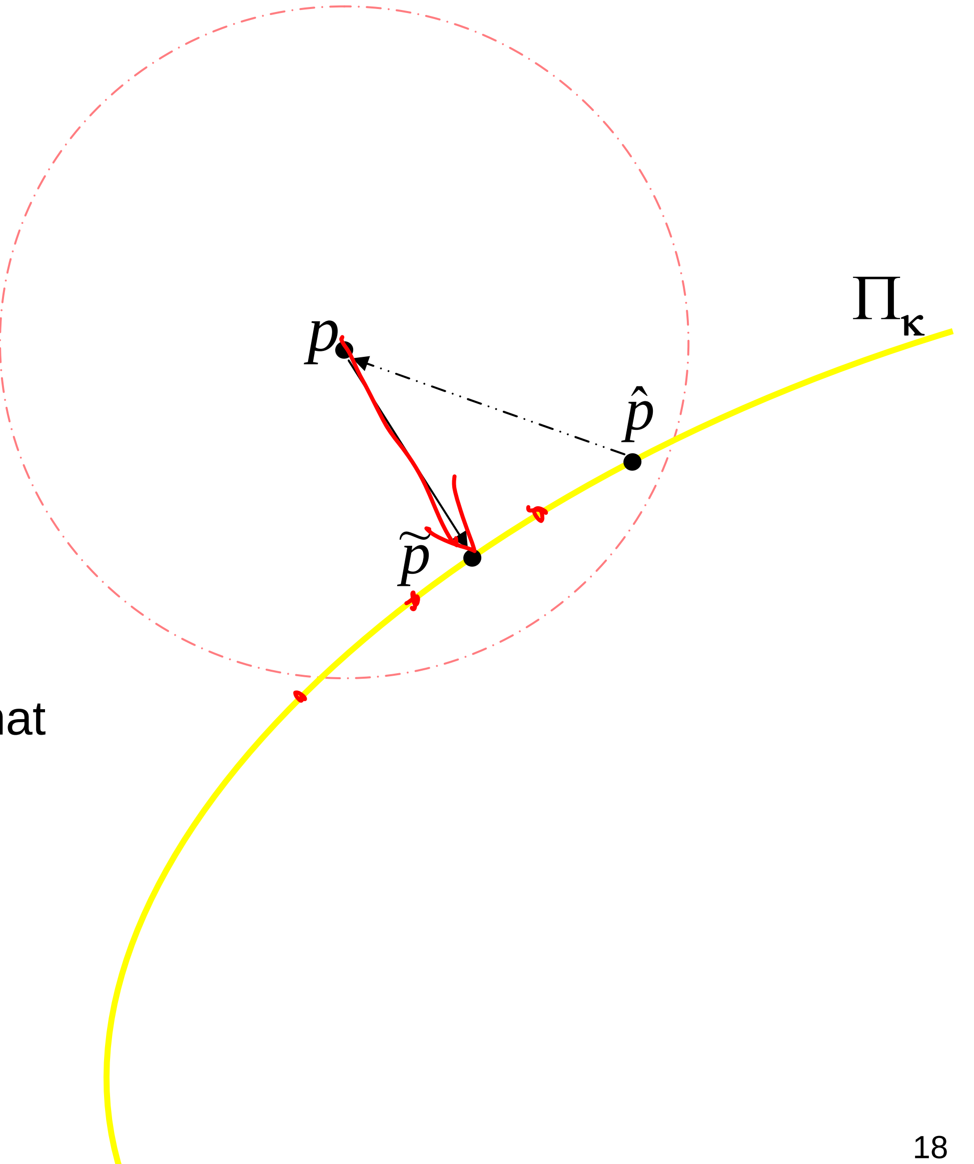
after formulating the AIF of p within ε

Stage I: Find the factorization manifold Π of the highest dimension s.t.

$$dist(P, \Pi_k) < \varepsilon$$

Stage II: Find/solve problem Q such that

$$\left\| p - \tilde{p} \right\| = \min_{q \in \Pi_k} \left\| p - q \right\|$$



Stage I: Identify the AIF manifold by a squarefree factorization

Example: $p_1^5\ p_2^3 p_3^3\ p_4$

$= (p_1 p_2 p_3 p_4)(p_1 p_2 p_3)(p_1 p_2 p_3)(p_1)(p_1)$ --- flat SFF

$= (p_4)^1 (1)^2 (p_2 p_3)^3 (1)^4 (p_1)^5$ --- staircase SFF

A new staircase SFF algorithm:

Input: $p \in \mathbb{C}[x], \quad \varepsilon > 0$

set $(u_0, v_0, w_0) = gcd_{\varepsilon}(p, p')$

for $k = 1, 2, \ldots,$ do

compute $(u_k, v_k, w_k) = gcd_{\varepsilon}(v_0,\ k v_0' - w_0)$

if $\sum_{j=1}^{k} j \cdot deg(u_j) = deg(p)$ then

set $l := k$, break, end if

Output : squarefree factors $u_1, \ldots, u_l$

$$p \approx u_1^1\, u_2^2 \cdots p_k^k \quad \Longrightarrow \quad a_0 (a_1 x + b_1)^{k_1} (a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n}$$

Stage II: Minimize the distance to the AIF manifold

$$
\begin{array}{rcl}
a_0(a_1x + b_1)^{k_1}(a_2x + b_2)^{k_2}\cdots(a_nx + b_n)^{k_n} & = & p \\
\alpha_1 a_1 + \beta_1 b_1 & = & \gamma_1 \\
\alpha_2 a_2 + \beta_2 b_2 & = & \gamma_2 \\
\ddots & \vdots & \vdots \\
\alpha_n a_n + \beta_n b_n & = & \gamma_n
\end{array}
$$

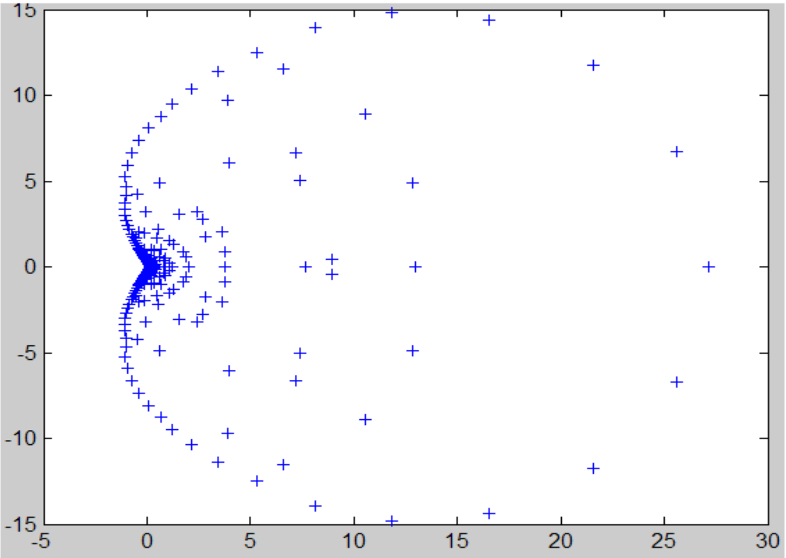
$$G(a_0, a_1, \ldots, a_n, b_1, \ldots, b_n) = d$$

Nonlinear least squares problem solved by the Gauss-Newton iteration

Example: For polynomial $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

with (inexact) coefficients in hardware precision

Exact factorization



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 ( x -      1.999999999999998 )^60
 ( x -      1.000000000000000 )^80
```

Summary:

- Factorizations are sensitive because polynomials form manifolds of positive codimensions in strata.
- An AIF can be formulated as an exact factorization of the nearest polynomial on a nearby manifold of the highest codimension.
- The AIF approximates the factorization of the (hidden) underlying polynomial from the perturbed data.
- The AIF can be computed by an (improved) algorithm in two stages.

Software is available in the package ApaTools (google apatools)