The *Approximate Irreducible Factorization* of a Univariate Polynomial. Revisited

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This talk revisits the work

An algorithm for accurate computation of multiple roots and multiplicities using floating point arithmetic even if the coefficients are perturbed.
A two-staged algorithm proposed in ISSAC 2003

For the polynomial

$$(x - 1)^{20}(x - 2)^{15}(x - 3)^{10}(x - 4)^5$$

with (inexact) coefficients in machine precision

Stage I results:

The backward error: 6.05 x 10^{-10}

Computed roots multiplicity:

1.0000000000000353 20
2.000000000000030904 15
3.0000000000176196 10
4.000000000109542 5

Stage II results:

The backward error: 6.16 x 10^{-16}

Computed roots multiplicity:

1.0000000000000000000000000000 20
1.9999999999999997 15
3.00000000000000011 10
3.999999999999999985 5

Software package MultRoot: Z. Zeng, ACM TOMS 2004
Example of a new method: For polynomial

\[(x - 1)^{80}(x - 2)^{60}(x - 3)^{40}(x - 4)^{20}\]

with (inexact) coefficients in hardware precision.
**Example:** For polynomial \((x - 1)^{80}(x - 2)^{60}(x - 3)^{40}(x - 4)^{20}\) with (inexact) coefficients in hardware precision

**Exact factorization**

**Approximate factorization:**

\[
\begin{align*}
\text{FACTORS} & \\
( x - 3.999999999999990 )^{20} \\
( x - 3.000000000000008 )^{40} \\
( x - 1.999999999999998 )^{60} \\
( x - 1.000000000000000 )^{80}
\end{align*}
\]

A significant advancement in robustness but *not really the point* for a revisit

**Question:** What problem are we really solving?
The Approximate Irreducible Factorization (also known as Root-finding) Problem:

Given

\[ p(x) = x^6 + 0.6667 x^5 - 2.333 x^4 - 1.333 x^3 + 1.667 x^2 + 0.6667 x - 0.3333 \]

Small perturbation

\[ \hat{p}(x) = (x-1)^2(x+1)^3\left(x - \frac{1}{3}\right)^1 \]

\[ = \hat{a}_0 (\hat{a}_1 x + \hat{b}_1)^2 (\hat{a}_2 x + \hat{b}_2)^3 (\hat{a}_3 x + \hat{b}_3)^1 \]

1. Match multiplicities

\[ |\hat{a}_i - \tilde{a}_i| + |\hat{b}_i - \tilde{b}_i| = O(\|p - \hat{p}\|) \]

2. \[ \hat{p}(x) = \tilde{a}_0 (\tilde{a}_1 x + \tilde{b}_1)^2 (\tilde{a}_2 x + \tilde{b}_2)^3 (\tilde{a}_3 x + \tilde{b}_3)^1 \]

\[ = 0.9999(1.0x - 1.00001)^2(1.0x + 1.000009)^3(1.0x - 0.3334)^1 \]

Conventional Factorization

\[ (x+1.0189+0.0034i)(x+1.019-0.00336i) \]
\[ (x+0.9621)(x-0.3332) \]
\[ (x-1+0.0144i)(x-1-0.0144i) \]

132 ) 3334 . 0 0 . 1 ( ) 000009 . 1 0 . 1 ( ) 00001 . 1 0 . 1 ( 9999 . 0

well-posed?
A well-posed problem: (Hadamard, 1923)

the solution satisfies

• existence
• uniqueness
• continuity w.r.t data

An ill-posed problem is \textit{infinitely} sensitive to perturbation

tiny perturbation $\Rightarrow$ huge error

A frontier in scientific computing

Though frequently needed in application, the adequate handling of such ill-posed … problems is hardly ever touched upon in numerical analysis textbooks.

--- Arnold Neumaier, SIAM Review
Challenge in solving ill-posed problems:

Can we recover the lost solution when the problem is inexact?

\[ P : \text{Data} \rightarrow \text{Solution} \]
Geometry of AIF (simplified view)

\[(x - t)^3 = -t^3 + (3t^2)x + (-3t)x^2 + x^3\]

\[F(t) = \begin{bmatrix} -t^3 \\ 3t^2 \\ -3t \end{bmatrix}\]

\[(x - u)^1(x - v)^2 = -uv^2 + (v^2 + 2uv)x + (-2v - u)x^2 + x^3\]

\[G(u, v) = \begin{bmatrix} -uv^2 \\ v^2 + 2uv \\ -2v - u \end{bmatrix}\]

Polynomials form (factorization) manifolds
Proposition 1: Polynomials

\[ c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m \]

\[ = a_0 (a_1 x + \hat{b}_1)^{k_1} (a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n} \]

form a differentiable manifold \( \Pi_{[k_1, \ldots, k_n]} \) of codimension

\[ \text{codim} \ (\Pi_{[k_1, \ldots, k_n]}) = m - n \]

---

dimension of the polynomial vector space (\( \geq \) polynomial degree)

Number of factors
Are ill-posed problems really sensitive?

Kahan: It is a misconception.

W. Kahan’s observation (1972)

- Problems form a “pejorative manifolds”

Plot of pejorative manifolds of degree 3 polynomials with multiple roots

- Ill-posedness: a tiny perturbation pushes the problem out of the manifold

- A problem is not sensitive at all if it stays on the manifold.
Stratification of factorization manifolds of degree 3 monic polynomials:

\[ \Pi(1,1,1) = \{ p(x) = (x - \alpha)^3(x - \beta)(x - \gamma) | \alpha \neq \beta \neq \gamma \} \]

\[ \Pi(1,2) = \{ p(x) = (x - \alpha)^3(x - \beta)^2 | \alpha \neq \beta \} \]

\[ \Pi(3) = \{ p(x) = (x - \alpha)^3 | \alpha \in C \} \]

\[ \overline{\Pi(3)} \subset \overline{\Pi(1,2)} \subset \overline{\Pi(1,1,1)} = \mathbb{C}^3 \]

Codimensions: 2 1 0

Factorization manifold stratification of degree 4 polynomials:

\[ \overline{\Pi(4)} \subset \overline{\Pi(2,2)} \subset \overline{\Pi(1,1,2)} \subset \overline{\Pi(1,1,1)} = \mathbb{C}^4 \]

Codimensions: 3 2 1 0
Factorization manifolds and their stratification

\[ \Pi_{[k_1 k_2 \cdots k_n]} = \left\{ a_0(a_1 x + b_1)^{k_1}(a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n} \mid a_i, b_i \in C, a_i b_j \neq a_j b_i, \forall i \neq j \right\} \]

\[ \subset C_m[x] = \left\{ c_0 + c_1 x + \cdots + c_m x^m \mid c_i \in C \right\} \]

**Proposition 3:**

\[ p \in \Pi_{[k_1 \cdots k_n]} \iff \text{codim}(\Pi_{[k_1 \cdots k_n]}) = \max \{ \text{codim}(\Pi) \mid \text{dist}(p, \Pi) = 0 \} \]
The approximate factorization of \( p \) is

- the exact factorization of \( \tilde{p} \)
- \( \tilde{p} \) lies in the nearby manifold \( \Pi \) of the highest codimension
- \( \hat{p} \) is the nearest polynomial on \( \Pi \) from \( p \)
A “three-strikes” principle for formulating an approximate irreducible factorization:

- **Backward nearness**: The AIF is the exact factorization of a nearby polynomial.
- **Maximum codimension**: The AIF is the exact factorization of a polynomial in the nearby factorization manifold of the highest codimension.
- **Minimum distance**: The AIF is the exact factorization of the nearest polynomial in the nearby factorization manifold of the highest codimension.

In comparison:

**Symbolic computation:**
- Backward nearness with distance = 0

**Numerical computation:**
- (straightforward) Backward nearness with minimal distance

⇒ Finding the AIF is (apparently) a well-posed problem
⇒ The AIF is a generalization of exact factorization.
Theorem 1

Assume $\|p - \hat{p}\|$ is small

Then, $\exists$ an interval $I$ and $\forall \varepsilon \in I$

$\exists$ a unique AIF within $\varepsilon$

$\tilde{p}(x) = \tilde{a}_0 (\tilde{a}_1 x + \tilde{b}_1)^{k_1} \cdots (\tilde{a}_n x + \tilde{b}_n)^{k_n} \in \Pi_k$

such that

$\|(\tilde{a}_i x + \tilde{b}_i) - (\hat{a}_i x + \hat{b}_i)\| = O(\|p - \hat{p}\|)$

Moreover, the AIF is continuous w.r.t. $p$

$Lipchitz ?$
The two-staged algorithm
after formulating the AIF of $p$ within $\varepsilon$

**Stage I:** Find the factorization manifold $\Pi$ of the highest dimension s.t.

$$\text{dist}(P, \Pi_k) < \varepsilon$$

**Stage II:** Find/solve problem $Q$ such that

$$\|p - \hat{p}\| = \min_{q \in \Pi_k} \|p - q\|$$
Stage I: Identify the AIF manifold by a squarefree factorization

Example:

\[ p_1^5 p_2^3 p_3^3 p_4 \]

\[ = (p_1 p_2 p_3 p_4) (p_1 p_2 p_3) (p_1 p_2 p_3) (p_1) (p_1) \]

\[ = (p_4)^1 (1)^2 (p_2 p_3)^3 (1)^4 (p_1)^5 \]

--- flat SFF

--- staircase SFF

A new staircase SFF algorithm:

\[
\begin{align*}
\text{Input: } & p \in \mathbb{C}[x], \ \varepsilon > 0 \\
& \text{set } (u_0, v_0, w_0) = \gcd_{\varepsilon}(p, p') \\
& \text{for } k = 1, 2, \ldots, \text{ do} \\
& \quad \text{compute } (u_k, v_k, w_k) = \gcd_{\varepsilon}(v_0, kv_0' - w_0) \\
& \quad \text{if } \sum_{j=1}^{k} j \cdot \deg(u_j) = \deg(p) \text{ then} \\
& \quad \quad \text{set } l := k, \ \text{break}, \ \text{end if} \\
\text{Output: } & \text{squarefree factors } u_1, \ldots, u_l
\end{align*}
\]

\[ p \approx u_1^1 u_2^2 \cdots p_k^k \quad \rightarrow \quad a_0(a_1x + b_1)^{k_1} (a_2x + b_2)^{k_2} \cdots (a_nx + b_n)^{k_n} \]
Stage II: Minimize the distance to the AIF manifold

\[ a_0 (a_1 x + b_1)^{k_1} (a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n} = p \]
\[ \alpha_1 a_1 + \beta_1 b_1 = \gamma_1 \]
\[ \alpha_2 a_2 + \beta_2 b_2 = \gamma_2 \]
\[ \vdots \]
\[ \alpha_n a_n + \beta_n b_n = \gamma_n \]

\[ G(a_0, a_1, \ldots, a_n, b_1, \ldots, b_n) = d \]

Nonlinear least squares problem solved by the Gauss-Newton iteration
**Example:** For polynomial \((x - 1)^{80}(x - 2)^{60}(x - 3)^{40}(x - 4)^{20}\)

with (inexact) coefficients in hardware precision

**Exact factorization**

![Graph of exact factorization]

**Approximate factorization:**

\[
\text{>> } [F, \text{res}, \text{fcnd}] = \text{uvFactor}(f, 1e-10, 1);
\]

- THE CONDITION NUMBER: 914.329
- THE BACKWARD ERROR: 5.71e-015
- THE ESTIMATED FORWARD ROOT ERROR: 1.04e-011

**FACTORS**

\[
\begin{align*}
(x - 3.999999999999990)^{20} \\
(x - 3.000000000000008)^{40} \\
(x - 1.999999999999998)^{60} \\
(x - 1.000000000000000)^{80}
\end{align*}
\]
Summary:

- Factorizations are sensitive because polynomials form manifolds of positive codimensions in strata.

- An AIF can be formulated as an exact factorization of the nearest polynomial on a nearby manifold of the highest codimension.

- The AIF approximates the factorization of the (hidden) underlying polynomial from the perturbed data.

- The AIF can be computed by an (improved) algorithm in two stages.

Software is available in the package ApaTools (google apatools)