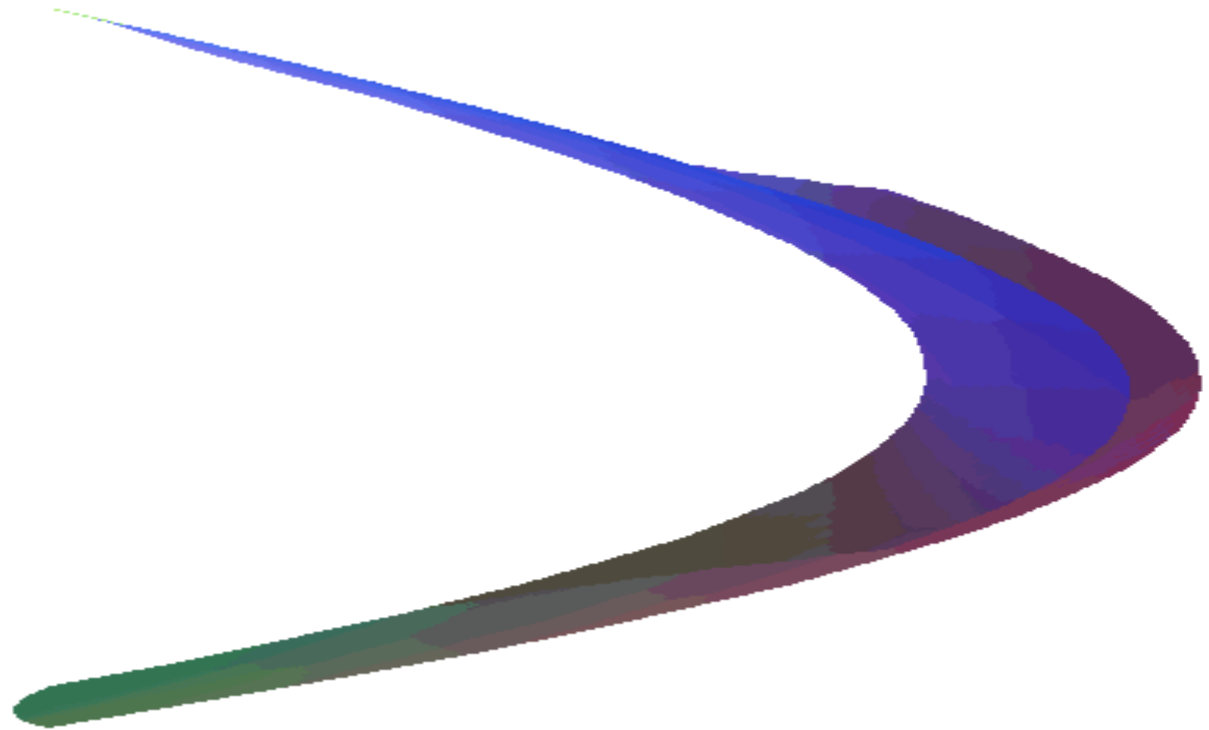


*The Approximate Irreducible Factorization  
of a Univariate Polynomial. Revisited*

Zhonggang Zeng  
Northeastern Illinois University



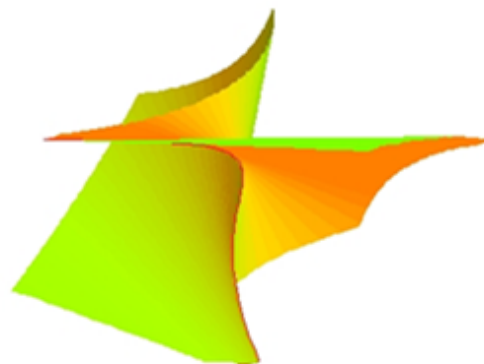
ISSAC 2009, KIAS, Seoul, July 31, 2009

(supported in part by NSF under Grant DMS-0715137)

This talk revisits the work

## Computing Multiple Roots of Inexact polynomials

Zhonggang Zeng



August 4, 2003 --- ISSAC'03

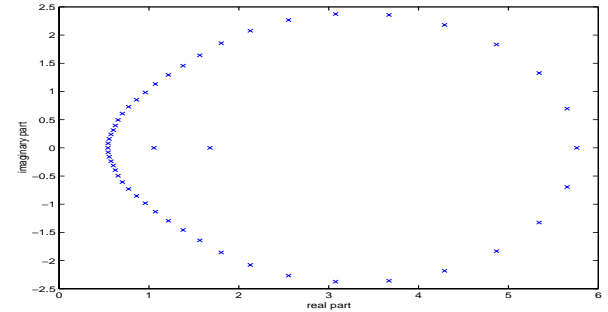
An algorithm for accurate computation of multiple roots and multiplicities using floating point arithmetic even if the coefficients are perturbed.

## A two-staged algorithm proposed in ISSAC 2003

For the polynomial

$$(x - 1)^{20} (x - 2)^{15} (x - 3)^{10} (x - 4)^5$$

with (inexact) coefficients in machine precision



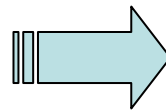
Stage I results:

The backward error:  $6.05 \times 10^{-10}$

Computed roots                      multiplicities

1.0000000000000000353  
2.000000000000000030904  
3.0000000000000000176196  
4.0000000000000000109542

20  
15  
10  
5



Stage II results:

The backward error:  $6.16 \times 10^{-16}$

Computed roots                      multiplicities

1.0000000000000000  
1.9999999999999997  
3.000000000000000011  
3.99999999999999985

20  
15  
10  
5

# Example of a new method: For polynomial

$$(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$$

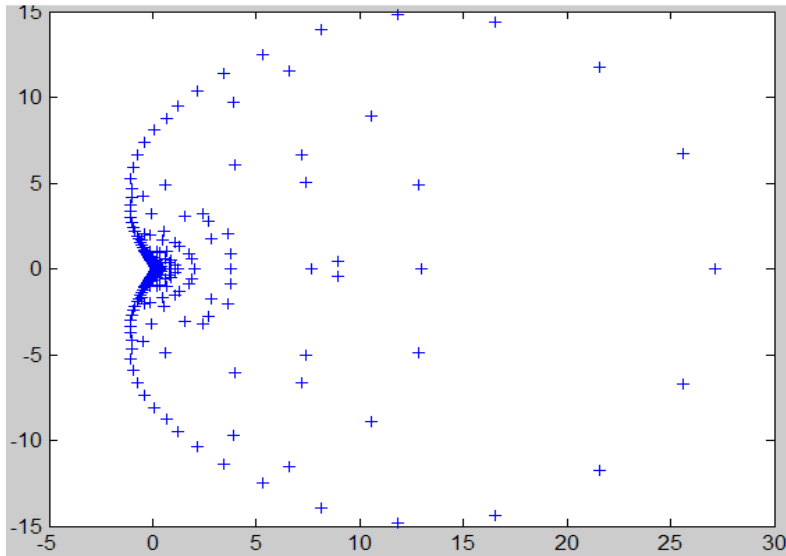
with (inexact) coefficients in hardware precision

```
> f := sort(expand((x-1.0)^80 * (x-2.0)^60 * (x-3.0)^40 * (x-4.0)^20));  
f = x200 - 400.0 x199 + 79500.00000000000 x198 - 0.1046764000000000 108 x197 + 0.1027178650000000 1010 x196 - 0.8012644644000000 1011 x195 + 0.5175538483300000 1013 x194 - 0.2847132038635600 1015 x193 - 0.1361679811118828 1017 x192  
- 0.5753552587503490 1018 x191 + 0.2172296640983603 1020 x190 - 0.7410201570953511 1021 x189 + 0.2302044364798637 1023 x188 - 0.6558201477371193 1024 x187 + 0.1723473499419453 1026 x186 - 0.4199314770979406 1027 x185 + 0.9528354787083167 1028  
- 0.2021339172401425 1030 x183 + 0.40226264988316151 1031 x182 - 0.75303201057972224 1032 x181 + 0.1331094716958985 1034 x180 - 0.2224356683846493 1035 x179 + 0.3525493747157756 1036 x178 - 0.5306966848760367 1037 x177 + 0.7603061333864248 1038  
- 0.1038452039998072 1040 x175 + 0.1354319862026686 1041 x174 - 0.1688968077489734 1042 x173 + 0.2016827194092254 1043 x172 - 0.2308886727671548 1044 x171 + 0.2537027024933842 1045 x170 - 0.2678575039599613 1046 x169 + 0.2720042441005213 1047  
- 0.2659190938436811 1048 x167 + 0.2505005438729205 1049 x166 - 0.2275690219157232 1050 x165 + 0.1995269567173599 1051 x164 - 0.1689635410331437 1052 x163 + 0.1382892324282459 1053 x162 - 0.1094641572260056 1054 x161 + 0.8385183542619580 1054  
- 0.6219634176530907 1055 x159 + 0.4469616573226471 1056 x158 - 0.3113567417215670 1057 x157 - 0.2103521439835924 1058 x156 - 0.1378933442295980 1059 x155 + 0.8774924975112167 1059 x154 - 0.5422914520826779 1060 x153 + 0.3256022272967521 1061  
- 0.1900094969044539 1062 x151 + 0.1078097408028366 1063 x150 - 0.5949641617502650 1063 x149 + 0.3194647503737332 1064 x148 - 0.1669541400327944 1065 x147 + 0.8494731901599281 1065 x146 - 0.420930055609420 1066 x145 + 0.203189462538380 1067  
- 0.9557423471287370 1067 x143 + 0.4381683825169602 1068 x142 - 0.1958434981777859 1069 x141 + 0.8535870525824515 1069 x140 - 0.3628730909318234 1070 x139 - 0.1504933038794333 1071 x138 - 0.6090335761233469 1071 x137 + 0.2405445246826672 1072  
- 0.9274045307037005 1072 x135 + 0.3490948970372923 1073 x134 - 0.12832049853161031 1074 x133 - 0.4608810373500992 1074 x132 - 0.1615370220314437 1075 x131 + 0.533261840033911 1075 x130 - 0.1833105022266985 1076 x129 + 0.6062775781651441 1076  
- 0.1938697912151828 1077 x127 + 0.6059976190572616 1077 x126 - 0.1851851770917562 1078 x125 - 0.5533066658373781 1078 x124 - 0.1616380576693199 1079 x123 + 0.4618974917346913 1079 x122 - 0.1290783348554147 1080 x121 + 0.3528263900759902 1080  
- 0.9434241189094910 1080 x119 + 0.2467897921296386 1081 x118 - 0.6316197550043220 1081 x117 + 0.1581701201009148 1082 x116 - 0.3875817535926498 1082 x115 + 0.9293948763244466 1082 x114 - 0.2181028213901091 1083 x113 + 0.5009223717578203 1083  
- 0.1126029844728893 1084 x111 + 0.2477536394177674 1084 x110 - 0.5335797953477143 1084 x109 - 0.1124875780397008 1085 x108 - 0.2321403150978500 1085 x107 + 0.4689758972457755 1085 x106 - 0.9275040680715252 1085 x105 + 0.1795792394900048 1086  
- 0.3403920183398081 1086 x103 + 0.6316721873546134 1086 x102 - 0.1147619161241749 1087 x101 - 0.2041269140353387 1087 x100 - 0.3554678154483986 1087 x99 + 0.6060343312533552 1087 x98 - 0.101155330008518 1088 x97 + 0.1652992710130165 1088 x96  
- 0.2644465840457458 1088 x95 + 0.4341735947568824 1088 x94 - 0.6350312579247448 1088 x93 + 0.953128248464562 1088 x92 - 0.1400489459458021 1089 x91 + 0.2014302984610080 1089 x90 - 0.2835853394318078 1089 x89 + 0.3907857869463126 1089 x88  
- 0.5270719447944147 1089 x87 + 0.6957536491826673 1089 x86 - 0.8988184010576921 1089 x85 + 0.1136302035477672 1090 x84 - 0.1405702987063490 1090 x83 + 0.1701538833095815 1090 x82 - 0.2015149408768288 1090 x81 + 0.2334835213964186 1090 x80  
- 0.26448386178327272 1090 x79 + 0.2934002366177785 1090 x78 - 0.3181544526387304 1090 x77 + 0.3373988904699309 1090 x76 - 0.3498910782983117 1090 x75 + 0.3547796960203292 1090 x74 - 0.3516996722798237 1090 x73 + 0.3408165031539125 1090 x72  
- 0.3228124282674547 1090 x71 + 0.2988159038901516 1090 x70 - 0.2702844667784428 1090 x69 + 0.2388579010425462 1090 x68 - 0.2062023202904667 1090 x67 + 0.1738657737643227 1090 x66 - 0.1431623029594843 1090 x65 + 0.1150969322547514 1090 x64  
- 0.9033145609890347 1089 x63 + 0.6919465769203387 1089 x62 - 0.5172257878878230 1089 x61 + 0.3771958380646445 1089 x60 - 0.2683143636765449 1089 x59 + 0.1861278354133417 1089 x58 - 0.1258828855026437 1089 x57 + 0.8298559057728609 1088 x56  
- 0.5330983254071900 1088 x55 + 0.3336287218941421 1088 x54 - 0.2035320379726298 1088 x53 + 0.1206797705895871 1088 x52 - 0.6970873497791365 1087 x51 + 0.3918017531075667 1087 x50 - 0.2142027549849079 1087 x49 + 0.1138699937488907 1087 x48  
- 0.3883798182093077 1086 x47 + 0.2913937246435160 1086 x46 - 0.1440330829905747 1086 x45 + 0.6817958216199887 1085 x44 - 0.3131710765672233 1085 x43 + 0.1395205726198939 1085 x42 - 0.6025499083555971 1084 x41 + 0.2521511610530864 1084 x40  
- 0.1021781974590337 1084 x39 + 0.4007239835501728 1083 x38 - 0.1520036677418415 1083 x37 + 0.3573150868413136 1082 x36 - 0.1973718556994919 1082 x35 + 0.6746682163537744 1081 x34 - 0.2224200938487125 1081 x33 + 0.7066120460785640 1080 x32  
- 0.2161361190562389 1080 x31 + 0.6359233600765355 1079 x30 - 0.1797948084199121 1079 x29 + 0.4879523867450775 1078 x28 - 0.1269710212868924 1078 x27 + 0.3163878742656724 1077 x26 - 0.7539495360019225 1076 x25 + 0.1715711274413615 1076 x24  
- 0.3722589973184066 1075 x23 + 0.7687860684582410 1074 x22 - 0.1508405412886810 1074 x21 + 0.2808075057063664 1073 x20 - 0.4938301837360766 1072 x19 + 0.8201255200350290 1071 x18 - 0.1281789842022448 1071 x17 + 0.1879570693272762 1070 x16  
- 0.2576999765404188 1069 x15 + 0.329077313227619 1068 x14 - 0.3896623869463437 1067 x13 + 0.4256749115413366 1066 x12 - 0.4264838624263202 1065 x11 + 0.3891785836191415 1064 x10 - 0.3207938674162293 1063 x9 + 0.2364733406710157 1062 x8  
- 0.1539696734805133 1061 x7 + 0.8716826972530130 1059 x6 - 0.4205487144414666 1058 x5 + 0.1678677181214472 1057 x4 - 0.3529844006643665 1055 x3 + 0.1261349023419938 1054 x2 - 0.1977831229290269 1052 x + 0.1541167191654754 1050
```

**Example:** For polynomial  $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

with (inexact) coefficients in hardware precision

Exact factorization



Approximate factorization:

```
>> [F,res,fcnd] = uvFactor(f,1e-10,1);
```

```
THE CONDITION NUMBER: 914.329
```

```
THE BACKWARD ERROR: 5.71e-015
```

```
THE ESTIMATED FORWARD ROOT ERROR: 1.04e-011
```

FACTORS

```
( x - 3.9999999999999990 )^20  
( x - 3.0000000000000008 )^40  
( x - 1.9999999999999998 )^60  
( x - 1.0000000000000000 )^80
```

A significant advancement in robustness  
but *not really the point* for a revisit

Question: What problem are we really solving?

# The Approximate Irreducible Factorization (also known as Root-finding) Problem:

Given

$$p(x) = x^6 + .6667 x^5 - 2.333 x^4 - 1.333 x^3 + 1.667 x^2 + 0.6667 x - 0.3333$$

Conventional Factorization

$$(x+1.0189+0.0034i)(x+1.019-0.00336i) \\ (x+0.9621)(x-0.3332) \\ (x-1+0.0144i)(x-1-0.0144i)$$

AIF

well-posed?

$$\tilde{p}(x) = \tilde{a}_0 (\tilde{a}_1 x + \tilde{b}_1)^2 (\tilde{a}_2 x + \tilde{b}_2)^3 (\tilde{a}_3 x + \tilde{b}_3)^1 \\ = 0.9999(1.0x - 1.00001)^2 (1.0x + 1.000009)^3 (1.0x - 0.3334)^1$$

Small perturbation

$$\hat{p}(x) = (x-1)^2 (x+1)^3 (x-\frac{1}{3})^1 \\ = \hat{a}_0 (\hat{a}_1 x + \hat{b}_1)^2 (\hat{a}_2 x + \hat{b}_2)^3 (\hat{a}_3 x + \hat{b}_3)^1$$

1. Match multiplicities

$$2. |\hat{a}_i - \tilde{a}_i| + |\hat{b}_i - \tilde{b}_i| = O(\|p - \tilde{p}\|)$$

## A well-posed problem: (Hadamard, 1923)

the solution satisfies

- existence
- uniqueness
- continuity w.r.t data



An ill-posed problem is infinitely sensitive to perturbation

tiny perturbation → huge error

## A frontier in scientific computing

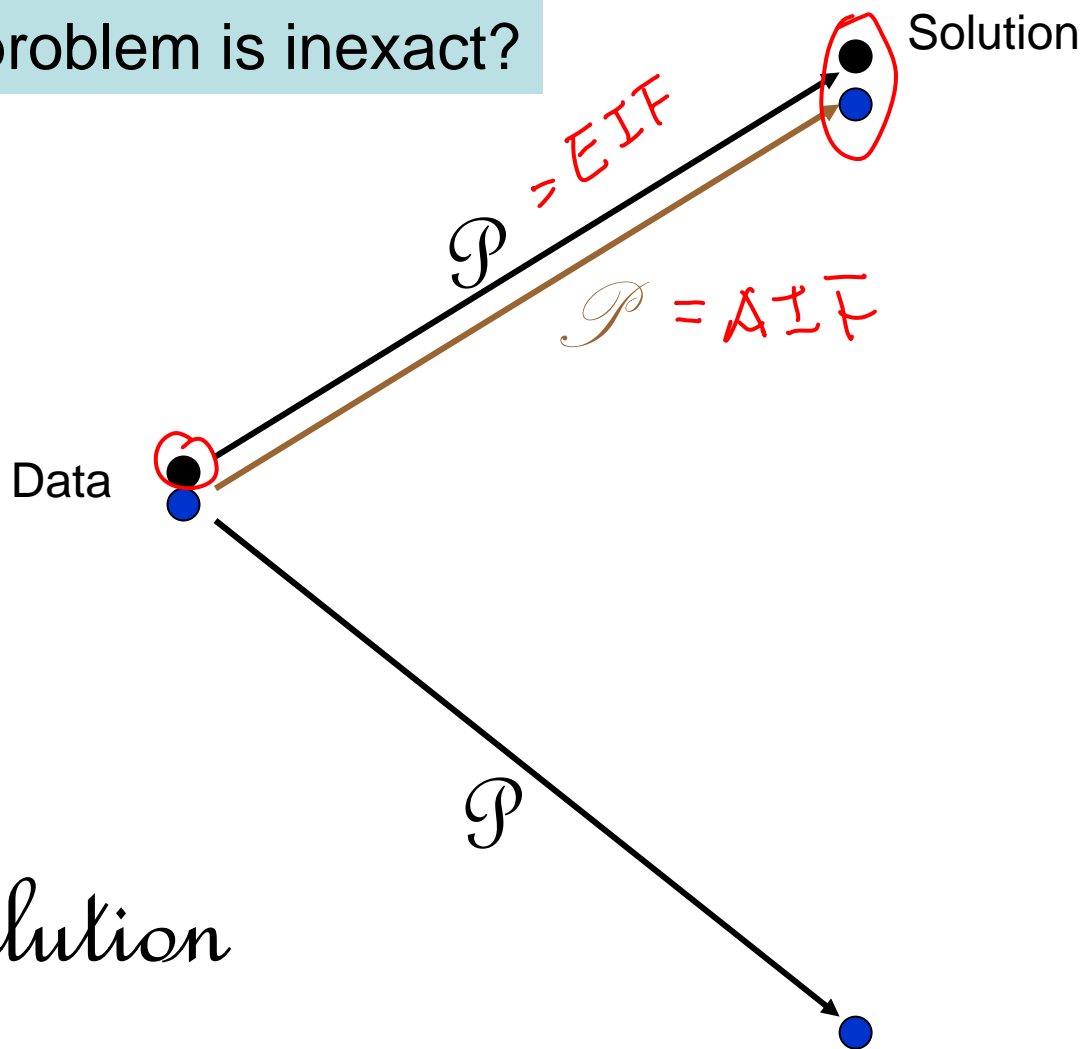
Though frequently needed in application, the adequate handling of such ill-posed ... problems is hardly ever touched upon in numerical analysis textbooks.



--- Arnold Neumaier, SIAM Review

# Challenge in solving ill-posed problems:

Can we recover the lost solution when the problem is inexact?



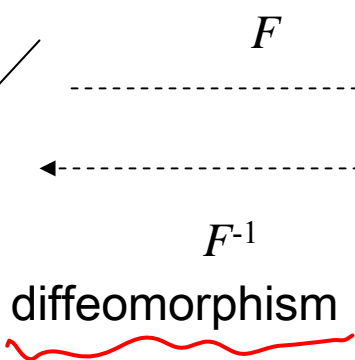
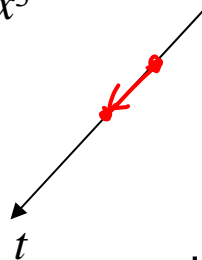
$$\mathcal{P} : \text{Data} \rightarrow \text{Solution}$$



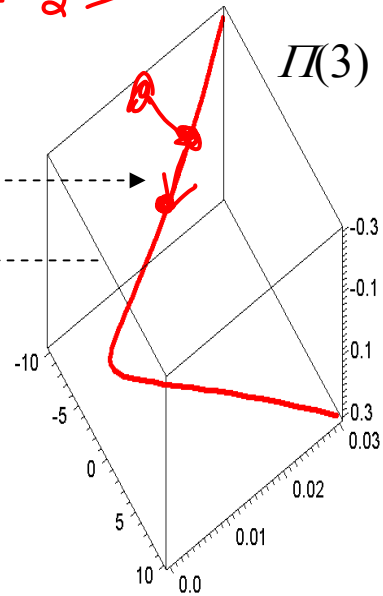
# Geometry of AIF (simplified view)

$$(x - t)^3 = -t^3 + (3t^2)x + (-3t)x^2 + x^3$$

$$F(t) = \begin{bmatrix} -t^3 \\ 3t^2 \\ -3t \end{bmatrix}$$

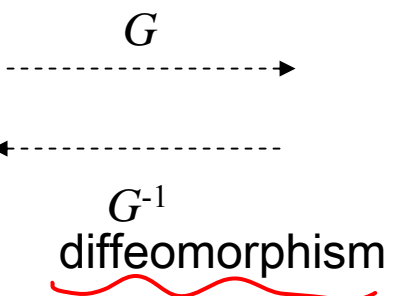
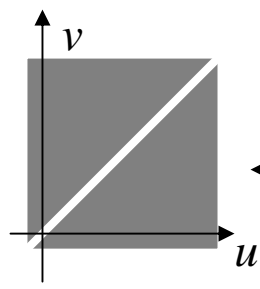


*codim = 2 > 0*

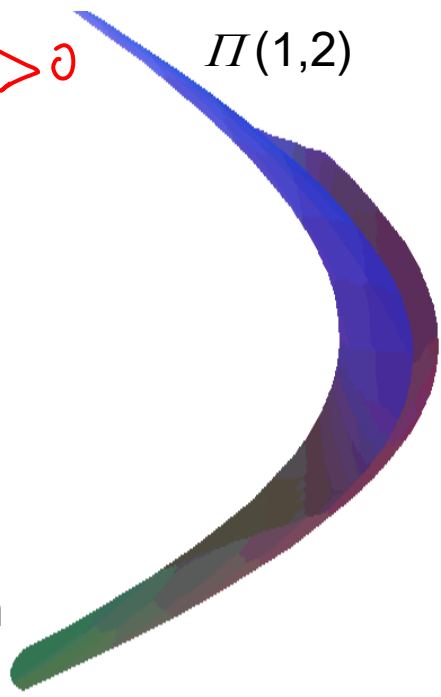


$$(x - u)^1 (x - v)^2 = -uv^2 + (v^2 + 2uv)x + (-2v - u)x^2 + x^3$$

$$G(u, v) = \begin{bmatrix} -uv^2 \\ v^2 + 2uv \\ -2v - u \end{bmatrix}$$




*codim = 1 > 0*



Polynomials form (factorization) manifolds

## Proposition 1: Polynomials

$$c_0 + c_1x + c_2x^2 + \cdots + c_mx^m$$
$$= a_0(a_1x + \hat{b}_1)^{k_1}(a_2x + b_2)^{k_2} \cdots (a_nx + b_n)^{k_n}$$


form a differentiable manifold  $\Pi_{[k_1, \dots, k_n]}$  of codimension

$$\text{codim} (\Pi_{[k_1, \dots, k_n]}) = m - n$$

dimension of the  
polynomial vector space  
( $\geq$  polynomial degree)

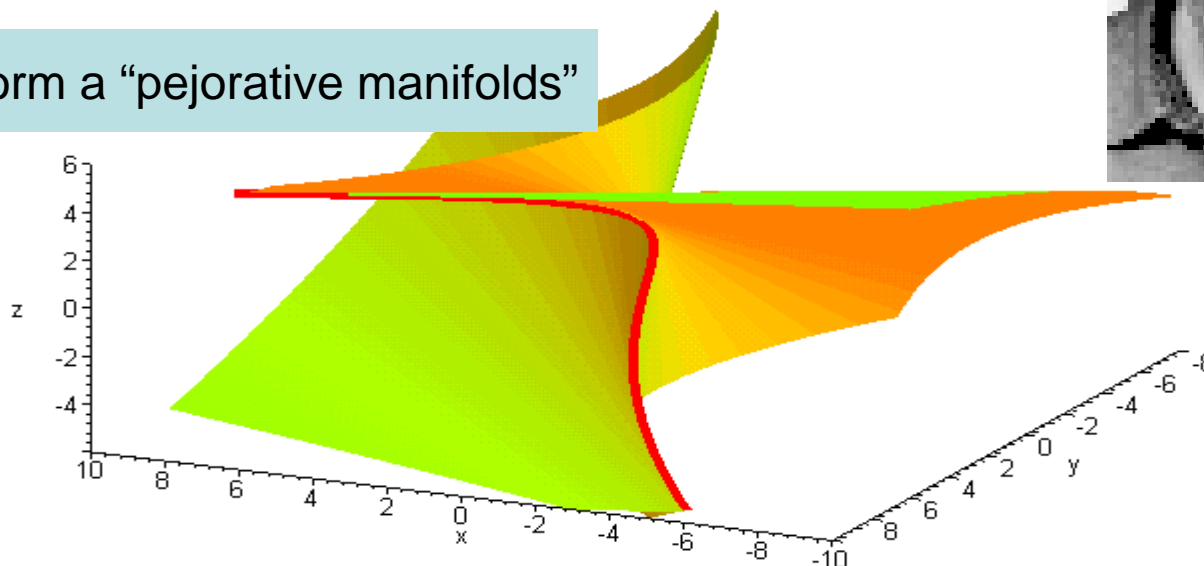
Number of factors

# Are ill-posed problems really sensitive?

Kahan: It is a misconception.

W. Kahan's observation (1972)

- Problems form a “pejorative manifolds”



Plot of pejorative manifolds of degree 3 polynomials with multiple roots



- Ill-posedness: a tiny perturbation pushes the problem out of the manifold

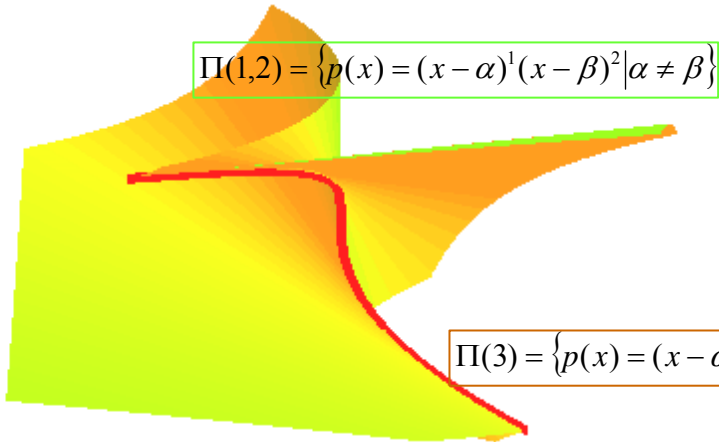
- A problem is not sensitive at all if it stays on the manifold.

# Stratification of factorization manifolds of degree 3 monic polynomials

$$\Pi(1,1,1) = \{p(x) = (x - \alpha)^1(x - \beta)^1(x - \gamma)^1 \mid \alpha \neq \beta \neq \gamma\}$$

$$\Pi(1,2) = \{p(x) = (x - \alpha)^1(x - \beta)^2 \mid \alpha \neq \beta\}$$

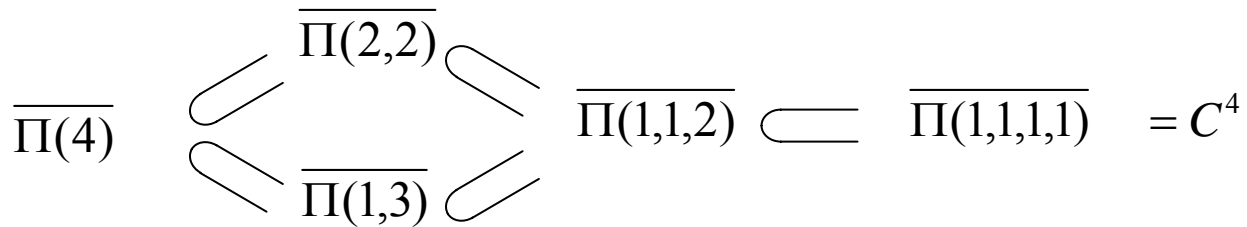
$$\Pi(3) = \{p(x) = (x - \alpha)^3 \mid \alpha \in \mathbb{C}\}$$



$$\overline{\Pi(3)} \subset \overline{\Pi(1,2)} \subset \overline{\Pi(1,1,1)} = \mathbb{C}^3$$

Codimensions: 2                      1                      0

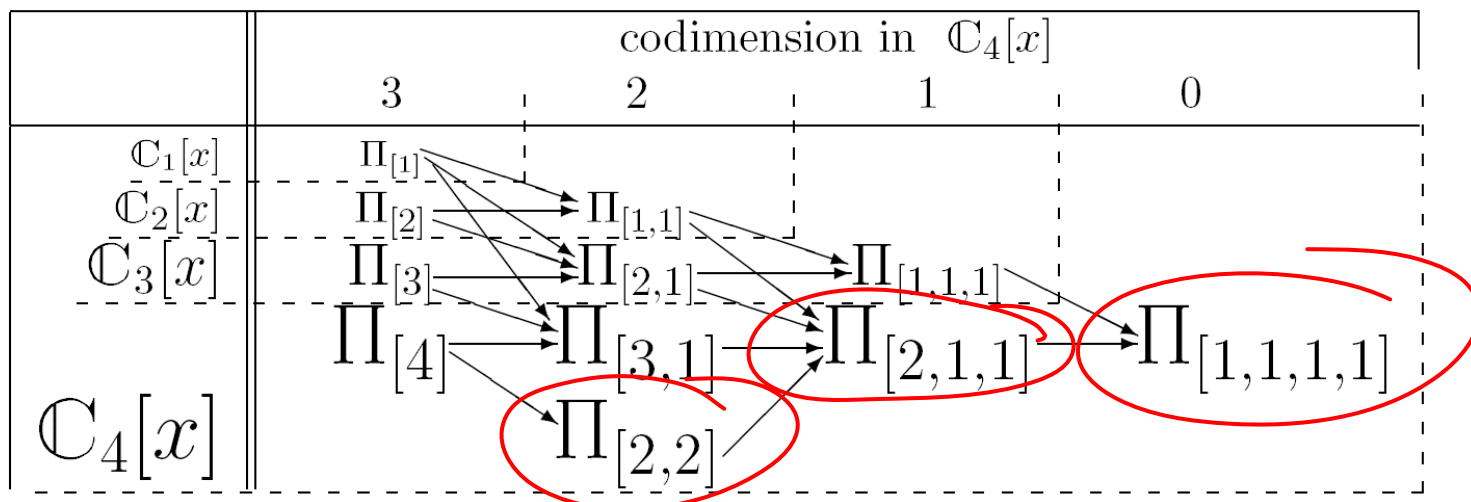
Factorization manifold stratification of degree 4 polynomials:



Codimensions: 3                      2                      1                      0

## Factorization manifolds and their stratification

$$\begin{aligned} \Pi_{[k_1 k_2 \dots k_n]} &= \left\{ a_0 (a_1 x + b_1)^{k_1} (a_2 x + b_2)^{k_2} \dots (a_n x + b_n)^{k_n} \mid a_i, b_i \in \mathbb{C}, a_i b_j \neq a_j b_i, \forall i \neq j \right\} \\ &\subset \mathbb{C}_m[x] = \left\{ c_0 + c_1 x + \dots + c_m x^m \mid c_i \in \mathbb{C} \right\} \end{aligned}$$



$$p \in \Pi_{[2,2]} \iff \text{dist}(p, \Pi_{[2,2]}) = \text{dist}(p, \Pi_{[2,1,1]}) = \text{dist}(p, \Pi_{[1,1,1,1]}) = 0$$

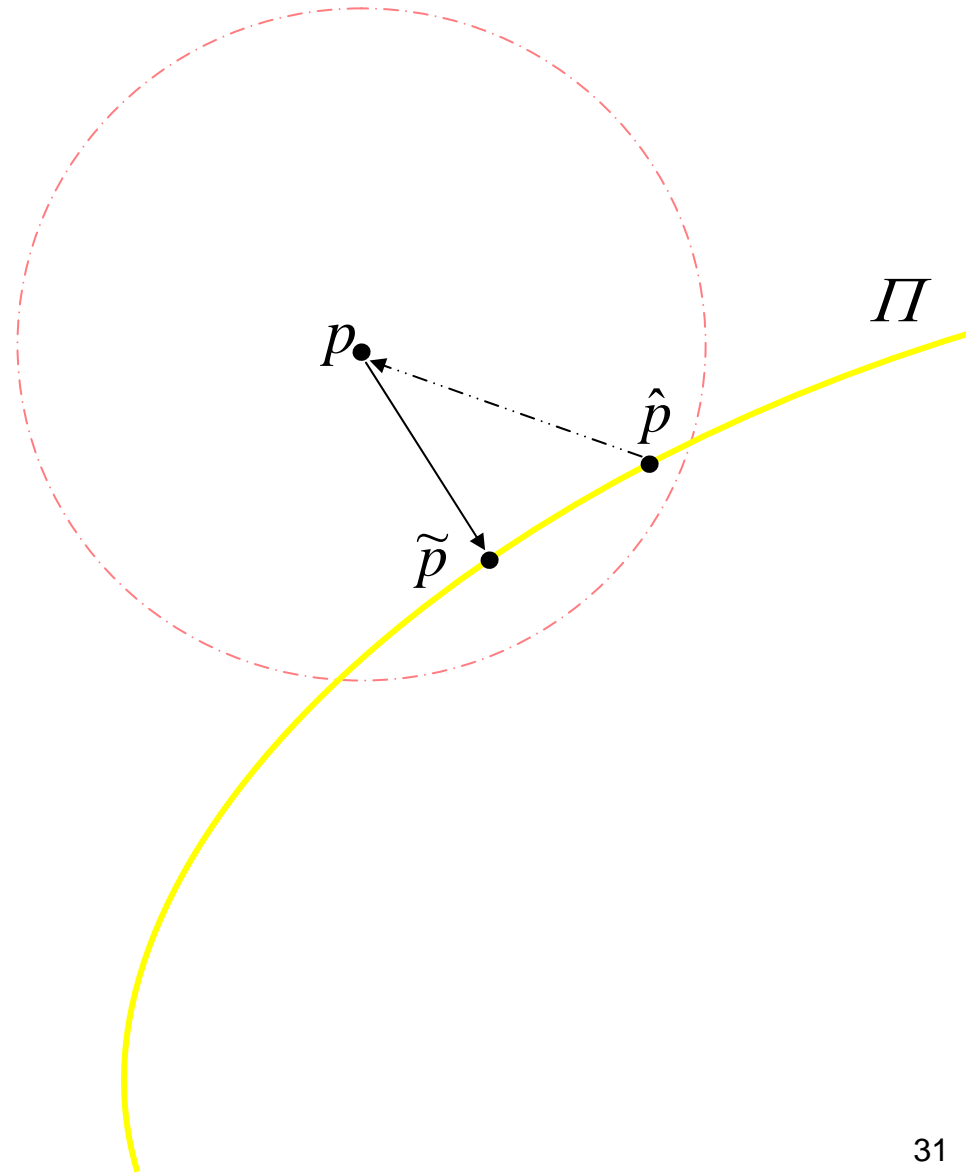
**Proposition 3:**  $p \in \Pi_{[k_1 \dots k_n]}$  if and only if

$$\text{codim}(\Pi_{[k_1 \dots k_n]}) = \max \{ \text{codim}(\Pi) \mid \text{dist}(p, \Pi) = 0 \}$$

## Formulation of the approximate irreducible factorization

The approximate factorization of  $p$  is

- the exact factorization of  $\tilde{p}$
- $\tilde{p}$  lies in the nearby manifold  $\Pi$  of the highest codimension
- $\tilde{p}$  is the nearest polynomial on  $\Pi$  from  $p$



A “three-strikes” principle for formulating an approximate irreducible factorization:

- **Backward nearness**: The AIF is the exact factorization of a nearby polynomial
- **Maximum codimension**: The AIF is the exact factorization of a polynomial in the nearby factorization manifold of the highest codimension.
- **Minimum distance**: The AIF is the exact factorization of the nearest polynomial in the nearby factorization manifold of the highest codimension.

In comparison:

Symbolic computation:

**Backward nearness** with  
distance = 0

Numerical computation:  
(straightforward)

**Backward nearness** with  
minimal distance

- Finding the AIF is (apparently) a well-posed problem
- The AIF is a generalization of exact factorization.

# Theorem 1

$$\hat{p}(x) = \hat{a}_0 (\hat{a}_1 x + \hat{b}_1)^{k_1} \cdots (\hat{a}_n x + \hat{b}_n)^{k_n} \in \Pi_k$$

Assume  $\|p - \hat{p}\|$  is small

Then,  $\exists$  an interval  $I$  and  $\forall \varepsilon \in I$

$\exists$  a unique AIF within  $\varepsilon$

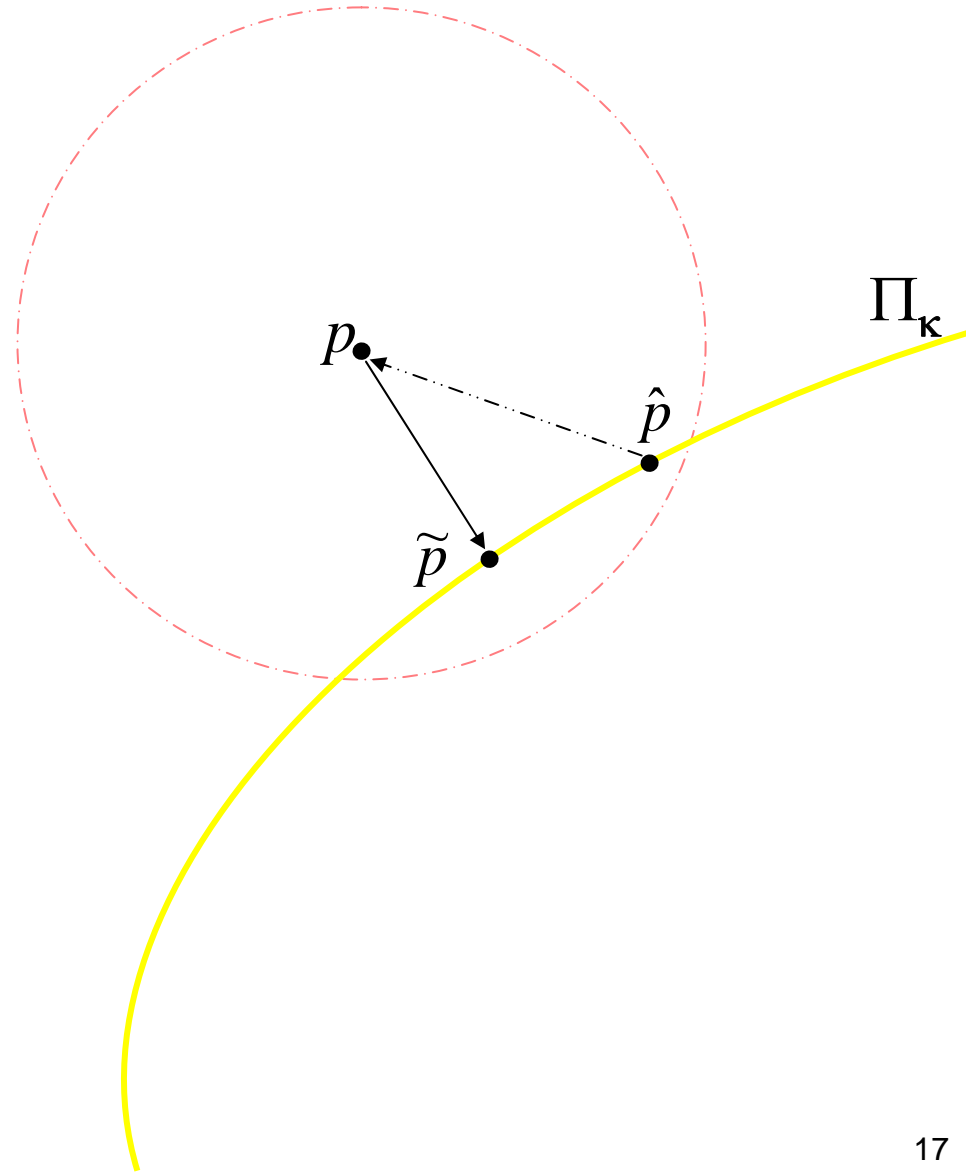
$$\tilde{p}(x) = \tilde{a}_0 (\tilde{a}_1 x + \tilde{b}_1)^{k_1} \cdots (\tilde{a}_n x + \tilde{b}_n)^{k_n} \in \Pi_k$$

such that

$$\|(\tilde{a}_i x + \tilde{b}_i) - (\hat{a}_i x + \hat{b}_i)\| = O(\|p - \hat{p}\|)$$

Moreover, the AIF is continuous w.r.t.  $p$

Lipshitz?







Stage I: Identify the AIF manifold by a squarefree factorization

Example:  $p_1^5 p_2^3 p_3^3 p_4$

$$= (p_1 p_2 p_3 p_4)(p_1 p_2 p_3)(p_1 p_2 p_3)(p_1)(p_1) \quad \text{--- flat SFF}$$

$$= (p_4)^1 (1)^2 (p_2 p_3)^3 (1)^4 (p_1)^5 \quad \text{--- staircase SFF}$$

A new staircase SFF algorithm:

Input:  $p \in \mathbb{C}[x], \quad \varepsilon > 0$   
 set  $(u_0, v_0, w_0) = \text{gcd}_\varepsilon(p, p')$   
 for  $k = 1, 2, \dots$ , do  
     compute  $(u_k, v_k, w_k) = \text{gcd}_\varepsilon(v_0, kv'_0 - w_0)$   
     if  $\sum_{j=1}^k j \cdot \text{deg}(u_j) = \text{deg}(p)$  then  
         set  $l := k$ , break, end if  
 Output : squarefree factors  $u_1, \dots, u_l$

$$p \approx u_1^1 u_2^2 \cdots p_k^k \quad \longrightarrow \quad a_0 (a_1 x + b_1)^{k_1} (a_2 x + b_2)^{k_2} \cdots (a_n x + b_n)^{k_n}$$

## Stage II: Minimize the distance to the AIF manifold

$$\begin{aligned} a_0(a_1x + b_1)^{k_1}(a_2x + b_2)^{k_2} \cdots (a_nx + b_n)^{k_n} &= p \\ \alpha_1 a_1 + \beta_1 b_1 &= \gamma_1 \\ & \\ \alpha_2 a_2 + \beta_2 b_2 &= \gamma_2 \\ & \\ & \vdots \\ & \vdots \\ \alpha_n a_n + \beta_n b_n &= \gamma_n \end{aligned}$$



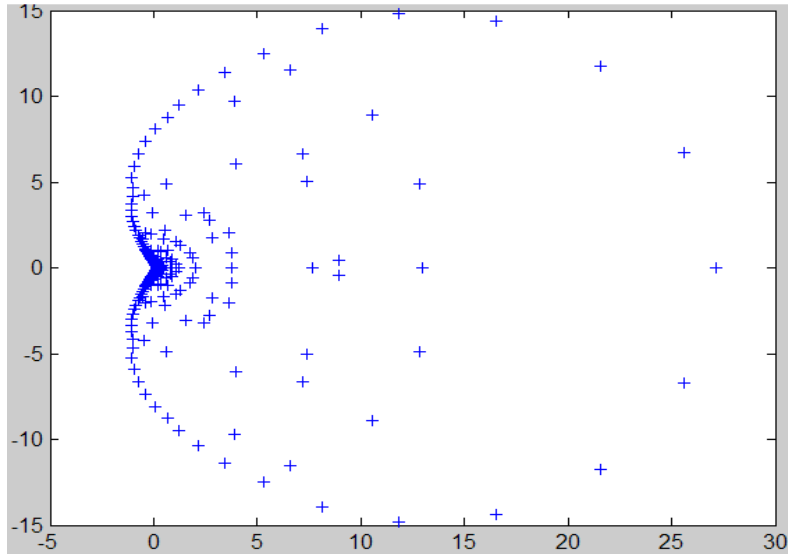
$$G(a_0, a_1, \dots, a_n, b_1, \dots, b_n) = d$$

Nonlinear least squares problem solved by the Gauss-Newton iteration

**Example:** For polynomial  $(x-1)^{80}(x-2)^{60}(x-3)^{40}(x-4)^{20}$

with (inexact) coefficients in hardware precision

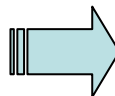
Exact factorization



Approximate factorization:

```
>> [F,res,fcnd] = uvFactor(f,1e-10,1);
```

```
THE CONDITION NUMBER:          914.329  
THE BACKWARD ERROR:           5.71e-015  
THE ESTIMATED FORWARD ROOT ERROR: 1.04e-011
```



**FACTORS**

```
( x -      3.9999999999999990 )^20  
( x -      3.0000000000000008 )^40  
( x -      1.9999999999999998 )^60  
( x -      1.0000000000000000 )^80
```

## Summary:

- Factorizations are sensitive because polynomials form manifolds of positive codimensions in strata.
- An AIF can be formulated as an exact factorization of the nearest polynomial on a nearby manifold of the highest codimension.
- The AIF approximates the factorization of the (hidden) underlying polynomial from the perturbed data.
- The AIF can be computed by an (improved) algorithm in two stages.

Software is available in the package `ApaTools`  
(`google apatools`)