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Isolating Real Roots of Real Polynomials

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VCA-bisection algorithm

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VCA (Vincent-Collins-Akritas) is a bisection algorithm to isolate the real roots of a polynomial $f \in \mathbb{R}[x]$.

- Starts with an interval I_0 that contains all roots (root bounds).
- For an Interval $I = (a, b)$ with midpoint m it considers the sign variations $\mathrm{var}(f, I)$ of the coefficients of $f_I := (1+x)^n \cdot f\left(\frac{ax+b}{1+x}\right)$:
  (i) Endpoints a and b are tested ($f(a) = 0$, $f(b) = 0$?).
  (ii) I is subdivided into intervals (a, m) and (m, b) if $\mathrm{var}(f, I) > 1$.
  (iii) I is **isolating** (contains exactly one root) if $\mathrm{var}(f, I) = 1$ and
  (iv) I is **discarded** (contains no root) if $\mathrm{var}(f, I) = 0$.

Works well for polynomials with integer (rational) coefficients.

Objective and Results

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- Deterministic, exact and efficient subdivision (bisection) algorithm to isolate the real roots of a polynomial $f(x) \in \mathbb{R}[x]$.
- Based on Descartes' Rule of Signs.
- Coefficients are assumed to be approximated to any specified error bound (bitstream coefficients).
- Algorithm comes in **two versions**:

  (i) square-free polynomial f

  (ii) not necessarily square-free f for which the number m of *distinct real roots* and $k := \deg(\gcd(f, f'))$ is known (from a precomputation step).

Prior Work

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It was suggested to replace the coefficients by small intervals and to execute the method (bisection algorithm) using interval arithmetic.

- First proposals (1997-2004) were incomplete; they all had to resort to exact arithmetic in the ring of coefficients for some input polynomials.
- Eigenwillig et. all (2005) showed that randomization leads to a complete bisection algorithm with no need for exact arithmetic.
- This randomized algorithm uses an adaptive precision management and runs on an interval polynomial in Bernstein representation. Subdivision points are chosen to stay away from roots of f.
- Upon this method, the *Geometric computing* group at MPII built and implemented several geometric algorithms:

Application

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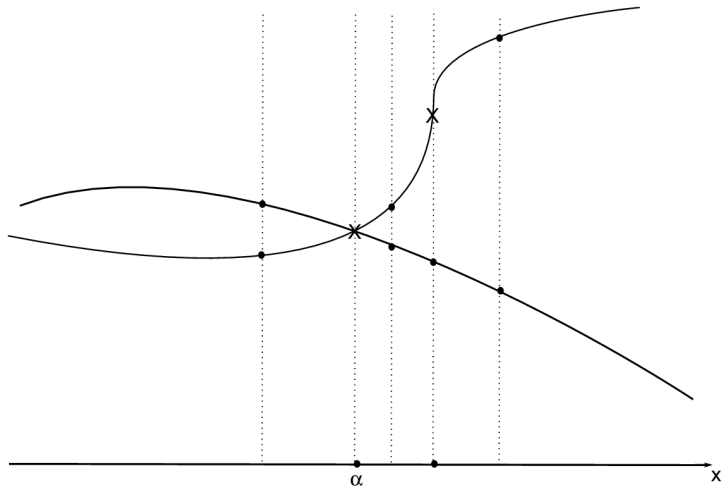
(i) Topology and arrangement computation of algebraic plane curves (EKW 2007, EK 2008). In EKW 07 an extension of the randomized Bitstream approach from 2005 for non square-free polynomials was also proposed.

(ii) Triangulation algorithm for implicitly defined algebraic surfaces (BKS 2008, BKS 2009).

- Lifting operation in cylindrical algebraic decomposition demands root isolation for a polynomial $f \in \mathbb{Q}(\alpha)[x]$ where α is algebraic.
- The two versions of our algorithm addresses this problem.



Application

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- The randomized version of the bitstream approach has turned out a **key ingredient** for exact and efficient algorithms for topology computations of algebraic curves in 2D and surfaces in 3D.
- Experiments (in Hemmer et al. 2009) also show good performance for polynomials with large integer coefficients (compared to methods that work on the exact polynomial).
- The randomized bitstream root isolator has become a mature package and will be integrated into the software library CGAL.

The new algorithm is **deterministic**, **easier** and its complexity analysis hints to the fact that it is at least as **efficient** as its randomized cousin.

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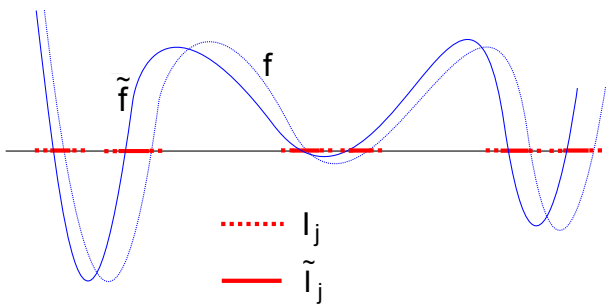
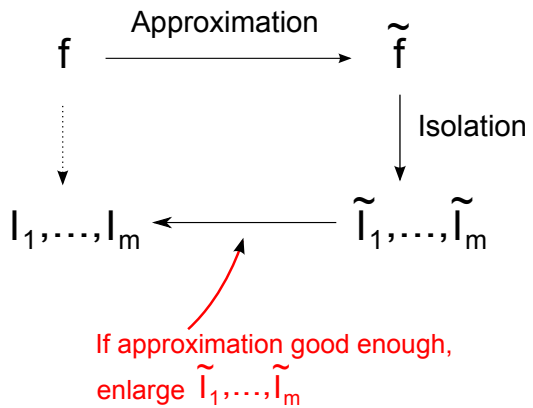
Main Idea

- For a bitstream polynomial $f \in \mathbb{R}[x]$, choose an arbitrary approximation $\tilde{f}$ of f.
- Determine isolating intervals $\tilde{I}_1, \ldots, \tilde{I}_m$ for the real roots of $\tilde{f}$.
- Enlarge the intervals $\tilde{I}_j$ to obtain isolating intervals $I_1, \ldots, I_m$ for the real roots of f.
- Success depends on whether the approximation was good enough.
  (i) real roots should stay real and imaginary ones imaginary.
  (ii) need some facts about how much the roots of f and $\tilde{f}$ differ.





Basics

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Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with bitstream coefficients $a_i \in \mathbb{R}$. Its roots are denoted $\Gamma := \{z_1, \ldots, z_n\} \subset \mathbb{C}$.

(i) The modulus of any root of f is bounded by

$$B := 1 + \max\left\{ \frac{|a_i|}{|a_n|}; 0 \leq i \leq n \right\}.$$

Instead of f we can consider $f(4B(x - \frac{1}{2}))$, whose roots are contained in a disc $\Delta \subset \mathbb{C}$ of radius $1/4$, centered at $1/2 + 0 \cdot i$.

(ii) In the following steps assume that $\Gamma \subset \Delta$. For a *real* root $z \in \Gamma$ we define $\sigma(z)$ as the distance of z to the nearest distinct root of f. For all other roots we define $\sigma(z)$ as the distance of z to the real axes. We denote $\sigma_f := \min_i \sigma(z_i)$ the **separation** of f.

Root Perturbation Bounds

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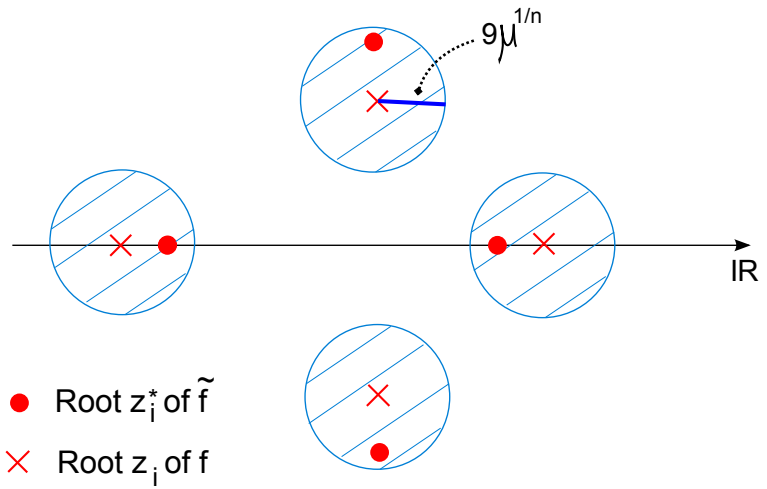
Consider an arbitrary, relative μ-approximation, $\mu < 2^{-7n}$, $\tilde{f} \in \mathbb{Q}[x]$ of f, that is $|f - \tilde{f}|_\infty < \mu |f|_\infty$, where we consider max-norm. We denote $z_1, \ldots, z_n$ and $z_1^*, \ldots, z_n^*$ the roots of f and $\tilde{f}$, respectively.

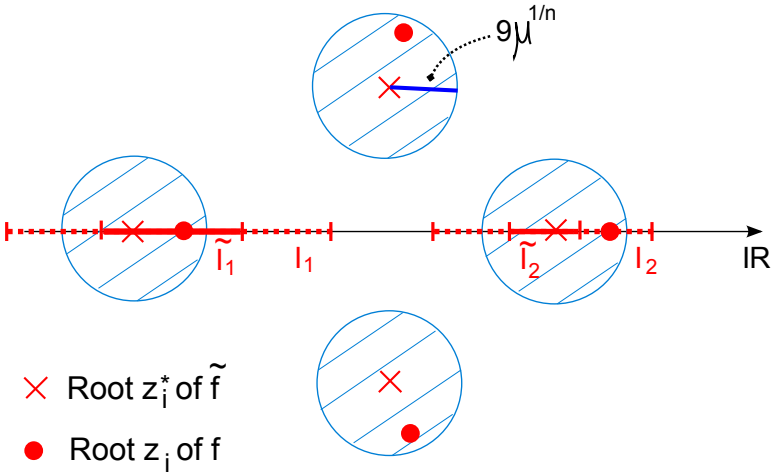
(iii) Due to a result of Schönhage (1985) we have $|z_i - z_i^*| < \rho := 9\sqrt[n]{\mu}$, up to permutation of the indices of the z_i^*.



How small must μ be

Our method can only be successful if the approximation was good enough, that is, μ is chosen small enough.

- For $\mu < \mu_0 := \min((\sigma_{\tilde{f}}/9)^n, 2^{-7n})$ the real roots stay real and imaginary roots stay imaginary when passing from $\tilde{f}$ to f.
- Assume that $\mu < \mu_0$ and that all $\tilde{I}_j$ are separated from each other by at least 2ρ. Then the intervals I_j, obtained by enlarging $\tilde{I}_j$ by ρ on both sides, are isolating for the real roots of f.



Estimating the separation

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- In order to use this observation we need a method to estimate $\sigma_{\tilde{f}}$.
- In particular, we need an dynamic estimate to check early whether μ was chosen sufficiently small?

We consider a slight modification of the well known VCA-bisection-algorithm (Vincent-Collins-Akritas), based on Descartes' Rule of Sign:

In contrast to its initial formulation it provides additional information about the separation σ_h of a polynomial $h \in \mathbb{R}[x]$ in terms of the minimal length $w_{\min}$ of an interval during the subdivision process.

VCA gives no Information about σ

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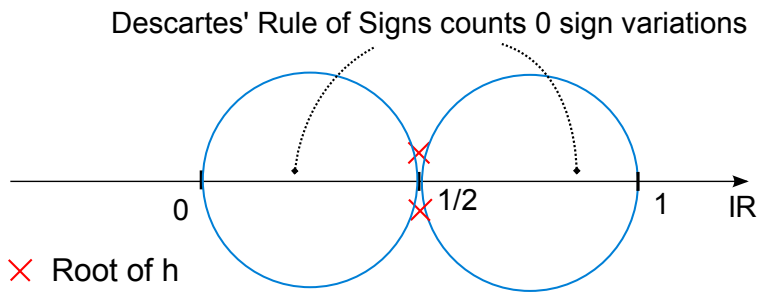
- Consider the polynomial
  $h(x) = 4x^2 + 4x + 1 + \delta^2 = (2x - 1 - i\delta)(2x - 1 + i\delta)$ with $\delta \approx 0$.
- h has a pair of conjugate complex roots at $1/2 \pm i\delta/2$ and hence separation $\delta/2$.
- However, the application of the VCA-bisection algorithm on the initial interval $(0, 1)$ terminates with the pair of intervals $(0, 1/2)$ and $(1/2, 1)$.



Descartes$^+$

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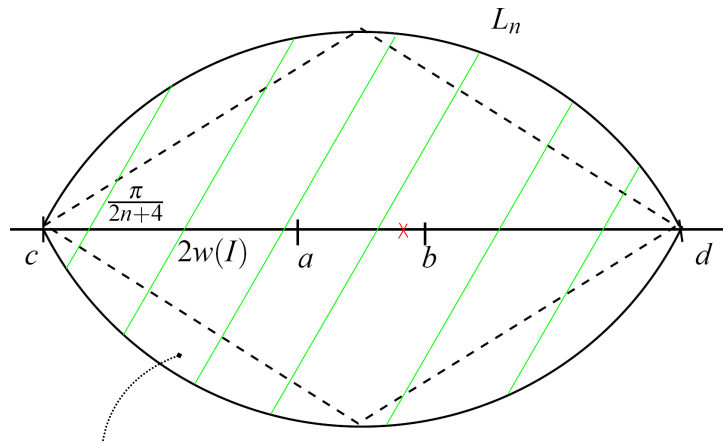
- VCA considers the sign variations $\mathrm{var}(h, I)$ on an interval I. I is subdivided if $\mathrm{var}(h, I) > 1$.
- Our extension *Descartes*$^+$ considers the sign variations on $I^+ := (a - 2w(I), b + 2w(I))$ and subdivides I if $\mathrm{var}(h, I^+) > 1$.
- *Descartes*$^+$ **splits no** interval of length $\leq \sigma_h/5$ and **refines at least one** interval to a length $< n\sigma_h$.
- Based on a generalization of the one and two-circle theorem due to Obreshkoff (1963).



Descartes' Rule of Signs considers all roots within L_n.
If $\mathrm{var}(h, I^+)=1$ then L_n contains exactly one root.

Algorithm for known σ_f

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If σ_f **is known** we choose $\mu < \min((\sigma_f/108)^n, 2^{-7n})$. Then:

(i) $\sigma_{\tilde{f}} > 90\sqrt[n]{\mu} = 5\rho$, thus real roots stay real and imaginary roots stay imaginary when passing from $\tilde{f}$ to f.

(ii) *Descartes*$^+$ splits no interval of length $\leq 2\rho < \sigma_{\tilde{f}}/5$. At least one interval has length $< 5n\rho$.

(iii) The isolating intervals $\tilde{I}_1, \ldots, \tilde{I}_k$ for the real roots of $\tilde{f}$ are always separated by an interval which contains no root. (Otherwise one of the intervals $\tilde{I}_j^+$ would contain two roots of $\tilde{f}$, thus $\mathrm{var}(f, I^+) > 1$.)

(iv) The enlarged intervals $I_1, \ldots, I_k$ are isolating intervals for the real roots of f.

Algorithm for unknown σ_f

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Run *Descartes*$^+$ on $\tilde{f}$ for a certain $\mu < 2^{-7n}$. If

(i) *Descartes*$^+$ produces no interval of length $\leq 2n\rho$, $\rho = 9\sqrt[n]{\mu}$, then $n\sigma_{\tilde{f}} > 2n\rho$, thus $\sigma_{\tilde{f}} > 2\rho$, and
(ii) two arbitrary isolating intervals for the roots of $\tilde{f}$ are always separated by at least $2n\rho$

then isolating intervals for the roots of f can be obtained by enlarging those for $\tilde{f}$ by ρ.

Idea: Run a *controlled version* of *Descartes*$^+$ on $\tilde{f}$: If *Descartes*$^+$ produces an interval of length $\leq 2n\rho$, replace μ by μ^2 and start over with a new, better approximation $\tilde{f}$ of f.

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Polynomials with Multiple Roots

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Now $f \in \mathbb{R}[x]$ is not necessary square-free.

- Number m of distinct real roots and $k := \deg(\gcd(f, f'))$ is known.
- The proposed algorithm $\mathit{Descartes}^{(m,k)}$ works under the following postcondition:

  (i) If f has exactly one multiple root it guarantees to isolate the real roots of f.

  (ii) In the case where f has more than one multiple root, it either outputs isolating intervals or returns a failure indicator. In the latter case, it provides the information that there exist more than one multiple root.

In the fiber computation for a cad (i) corresponds to a generic situation and (ii) to the non-generic case.

Treesize and Approximation

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The complexity analysis is mainly based on the following:

- In each subdivision level, the width of the subdivision tree, generated by *Descartes*$^+$ (*Descartes*$^{(m,k)}$), matches those of the VCA bisection algorithm up to a constant factor.
- For a polynomial f with coefficients bounded by 2^τ the algorithms *Descartes*$^+$ and *Descartes*$^{(m,k)}$ need coefficient approximations to $O(n(\log \frac{1}{\sigma_f} + \log n + \tau))$ bits after the binary point and
- have cost of $O(n^4(\log \frac{1}{\sigma_f} + \log n + \tau)^2)$ bit operations.

Conclusion and Outlook

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- We plan to implement our deterministic algorithm to see whether it is competitive with its randomized cousin.
- Generalization of the algorithm to other subdivision methods, in particular, continued fraction methods.
- Improvement of the precision requirement.

Thank you for your attention!

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Thank you for your attention!