Interoperating between Computer Algebra Systems: Computing Homology of Groups with Kenzo and GAP

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- Algebra: $H_i(G)$ are computed by means of *resolutions*.

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Question: could HAP and Kenzo cooperate in computations where homology of groups is needed?

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A *resolution* F_* for a group G is an acyclic chain complex of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} = \mathbb{Z} \longrightarrow 0$$

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Theorem

Let G be a group and F_*, F'_* two free resolutions of G. Then

$$H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F'_*) \cong H_n(K(G,1)) \quad \textit{for all } n \in \mathbb{N}$$

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Definition

Given a group G, the *homology groups* $H_n(G)$ are defined as $H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*)$, $n \in \mathbb{N}$, where F_* is any free resolution for G.

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For some particular cases, small (or minimal) resolutions can be directly constructed.

For instance, let $G = C_m$ with generator t. The resolution F_*

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

produces

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } \ i = 0 \\ \mathbb{Z}/m\mathbb{Z} & \text{if } \ i \text{ is odd} \\ 0 & \text{if } \ i \text{ is even}, i > 0 \end{cases}$$

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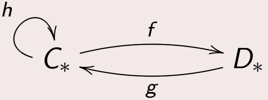
Definition

A *reduction* ρ between two chain complexes C_* and D_* (denoted by $\rho : C_* \Rrightarrow D_*$) is a triple $\rho = (f, g, h)$



satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
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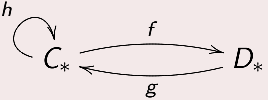

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If $C_* \Rrightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

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A *(strong chain) equivalence* ε between C_* and D_*, $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple $\varepsilon = (B_*, \rho, \rho')$ where B_* is a chain complex, $\rho : B_* \Rrightarrow C_*$ and $\rho' : B_* \Rrightarrow D_*$.

$$\begin{array}{ccccc} & & B_* & & \\ & \swarrow\!\!\swarrow & & \searrow\!\!\searrow & \\ C_* & & & & D_* \end{array} \qquad\qquad \begin{array}{ccccc} & & \frac{42}{30} & & \\ & \swarrow\!\!\swarrow & & \searrow\!\!\searrow & \\ \frac{14}{10} & & & & \frac{21}{15} \end{array}$$

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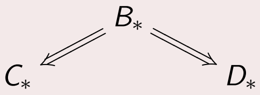

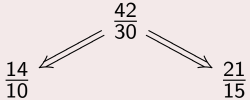

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An *object with effective homology* is a quadruple $(X, C_*(X), HC_*, \varepsilon)$ where HC_* is an effective chain complex and $\varepsilon : C_*(X) \Longleftarrow\!\!\Longrightarrow HC_*$.

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This implies that $H_n(X) \cong H_n(HC_*)$ for all n.

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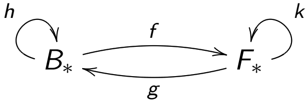
It is well known that there exists a morphism of chain complexes of $\mathbb{Z}G$-modules $f : B_* \to F_*$ which is a homotopy equivalence. An algorithm has been designed constructing the explicit expressions of f and the corresponding maps g, h and k



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Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$\overset{h}{\curvearrowright}\; \mathbb{Z} \otimes_{\mathbb{Z}G} B_* \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{Z} \otimes_{\mathbb{Z}G} F_* \;\overset{k}{\curvearrowleft}$$

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In order to obtain a strong chain equivalence we make use of the mapping cylinder construction.

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \overset{\rho'}{\Lleftarrow} \text{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$$

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Finally we observe that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(K(G,1))$. Moreover, if the initial resolution F_* is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_* \equiv E_*$ is effective.

Algorithm computing the effective homology of a group

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Algorithm

Input: a group G and a free resolution F_* of finite type with contracting homotopy.
Output: the effective homology of $K(G,1)$, that is, a (strong chain) equivalence $C_*(K(G,1)) \lll\!\!\Longleftrightarrow\!\!\ggg E_*$ where E_* is an effective chain complex.

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    - group
    - highest degree
    - list of ranks of each $\mathbb{Z}G$-module
    - boundary map
    - contracting homotopy

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Group:

```
<OMA>
     <OMS cd="group1" name="Group"/>
     <OMA>
          <OMS cd="permut1" name="Permutation"/>
          <OMI> 3</OMI>
          <OMI> 2</OMI>
          <OMI> 1</OMI>
     </OMA>
     <OMA>
          <OMS cd="permut1" name="Permutation"/>
          <OMI> 1</OMI>
          <OMI> 3</OMI>
          <OMI> 5</OMI>
          <OMI> 4</OMI>
          <OMI> 2</OMI>
     </OMA>
</OMA>
```

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Highest degree:

```
<OMI> 6</OMI>
```

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```
<OMI> 6</OMI>
```

List of ranks:

```
<OMA>
     <OMS cd="list1" name="list"/>
     <OMI> 1</OMI>
     <OMI> 3</OMI>
     <OMI> 6</OMI>
     <OMI> 10</OMI>
     <OMI> 15</OMI>
     <OMI> 20</OMI>
     <OMI> 26</OMI>
</OMA>
```

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The $\mathbb{Z}G$-boundary and the contracting homotopy are represented as lists containing the images of the generators of each module F_i. For instance, $F_1 = (\mathbb{Z}G)^3$ has three generators. The boundary of the first one is the combination $1 * (g_2, z_1) - 1 * (g_1, z_1)$, represented in OpenMath as:

```
<OMA><OMS cd="resolutions" name="zgcombination"/>
     <OMA><OMS cd="resolutions" name="zgterm"/>
          <OMI> 1</OMI>
          <OMA><OMS cd="resolutions" name="zggnrt"/>
               <OMI> 2</OMI>
               <OMI> 1</OMI>
          </OMA>
     </OMA>
     <OMA><OMS cd="resolutions" name="zgterm"/>
          <OMI> -1</OMI>
          <OMA><OMS cd="resolutions" name="zggnrt"/>
               <OMI> 1</OMI>
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          </OMA>
     </OMA>
</OMA>
```

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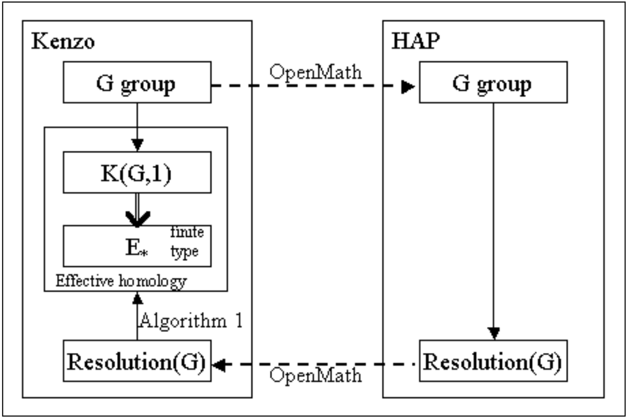

Figure: Communication between Kenzo and HAP

Applications and examples

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Computations with $K(G, n)'s$

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```
> (setf KC71 (K-G-1 C7))
[K82 Abelian-Simplicial-Group]
> (efhm KC71)
[K119 Homotopy-Equivalence K82 <= K109 => K76]
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```
> (setf KC72 (classifying-space KC71))
[K120 Abelian-Simplicial-Group]
> (efhm KC72)
[K259 Homotopy-Equivalence K120 <= K249 => K245]
> (homology KC72 3 6)
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/7Z
---done---
Homology in dimension 5 :
---done---
```

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Homology of 2-types

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Homology of 2-types

Let $G = C_3$, $A = \mathbb{Z}/3\mathbb{Z}$ with trivial G-action, and $[f] \in H^3(G, A) = \mathbb{Z}/3\mathbb{Z}$ a non-trivial cohomology class.

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```
> (setf K-C3-1 (K-Cm-n 3 1))
[K261 Abelian-Simplicial-Group]
> (setf chml-clss (chml-clss K-C3-1 3))
[K308 Cohomology-Class on K288 of degree 3]
> (setf tau (zp-whitehead 3 K-C3-1 chml-clss))
[K323 Fibration K261 -> K309]
> (setf x (fibration-total tau))
[K329 Kan-Simplicial-Set]
> (efhm x)
[K541 Homotopy-Equivalence K329 <= K531 => K527]
> (homology x 5)
Homology in dimension 5 :
Component Z/3Z
---done---
```