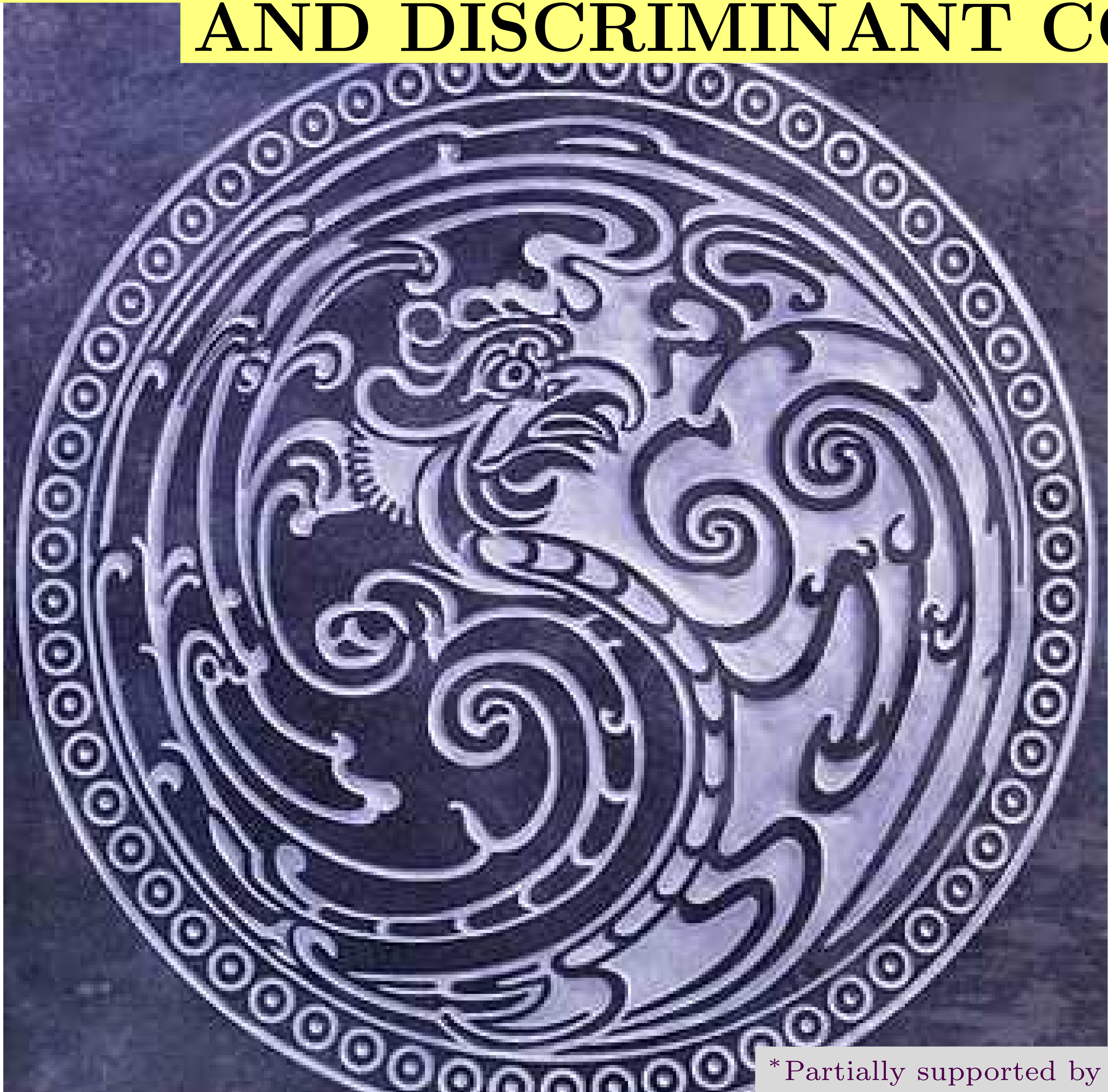
FASTER REAL FEASIBILITY AND DISCRIMINANT COMPLEXITY



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MAIN QUESTION

Can one compute, in time polynomial in the **sparse** *size, an integer that is, with high probability, the number of connected components of the real zero set of a random input polynomial?*

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We take a step toward a positive answer through our results...

WARM-UP...

Suppose you want to find the exact number of positive roots of

$$1 - 2x^{196418} + x^{317811} \ldots$$

THE RIGHT TOOL?

Suppose you want to find the exact number of positive roots of

$$1 - 2x^{196418} + x^{317811}...$$

Let's compare

Sturm Sequences and **Discriminant Chambers**...

STURM SEQUENCES...

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\begin{aligned}
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f_2 &:= -\mathrm{rem}(f_0/f_1) = -317811 + 242786x^{196418}
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...now count sign alternations in $(f_0(t), f_1(t), \ldots, f_{57}(t))$ for $t = 0$ and $t = +\infty$, and then subtract. (Should get **2** here.)

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Can we attain complexity polynomial in log(degree)?

YES!

THEOREM 1

[Bihan-Rojas-Stella] *Fix* n. *Then for any "honest"* n*-variate* $(n+2)$*-nomial* f, *one can decide* $Z_+(f) \stackrel{?}{=} \emptyset$ *in* $\mathbf{P}$.

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Note:

Input size:= # of bits needed to write monomial term expansion.

e.g., $\mathbf{Size}\left(a+b+cx_1^D x_2^D\right) = O(\log(a)+\log(b)+\log(c)+\log(D))$

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Note: All earlier algorithms (even much more general results of Basu, Gabrielov, and Zell) yield singly exponential time at best.

KEY TRICK FOR $n = 1$

Look at the graph of

$$f(x_1) := 1 - cx_1^d + x_1^D \quad (0 < d < D) \ldots$$

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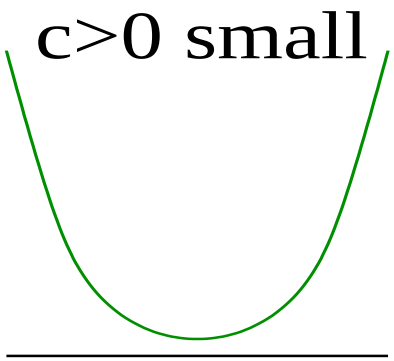

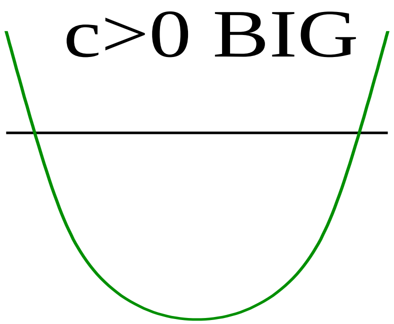

BASIC CALCULUS

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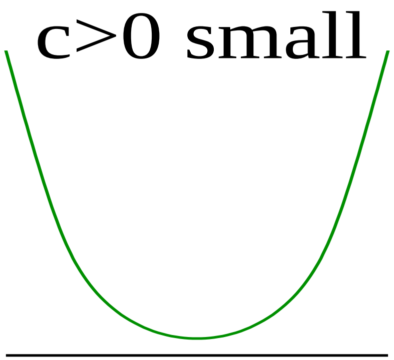

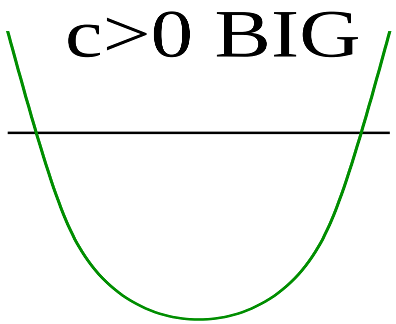

$f = f' = 0$ has a root $\zeta \in \mathbb{C} \setminus \{0\} \Longleftrightarrow [1, c\zeta^d, \zeta^D]^T$ is a right-null vector for $\begin{bmatrix} 1 & 1 & 1 \\ 0 & d & D \end{bmatrix}$...

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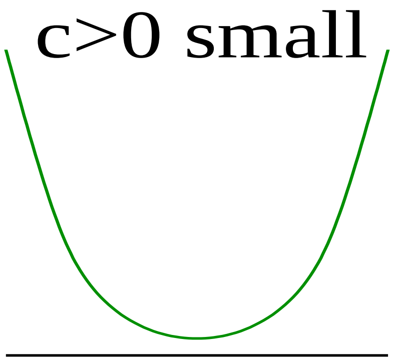
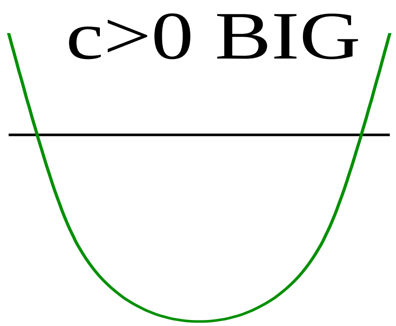




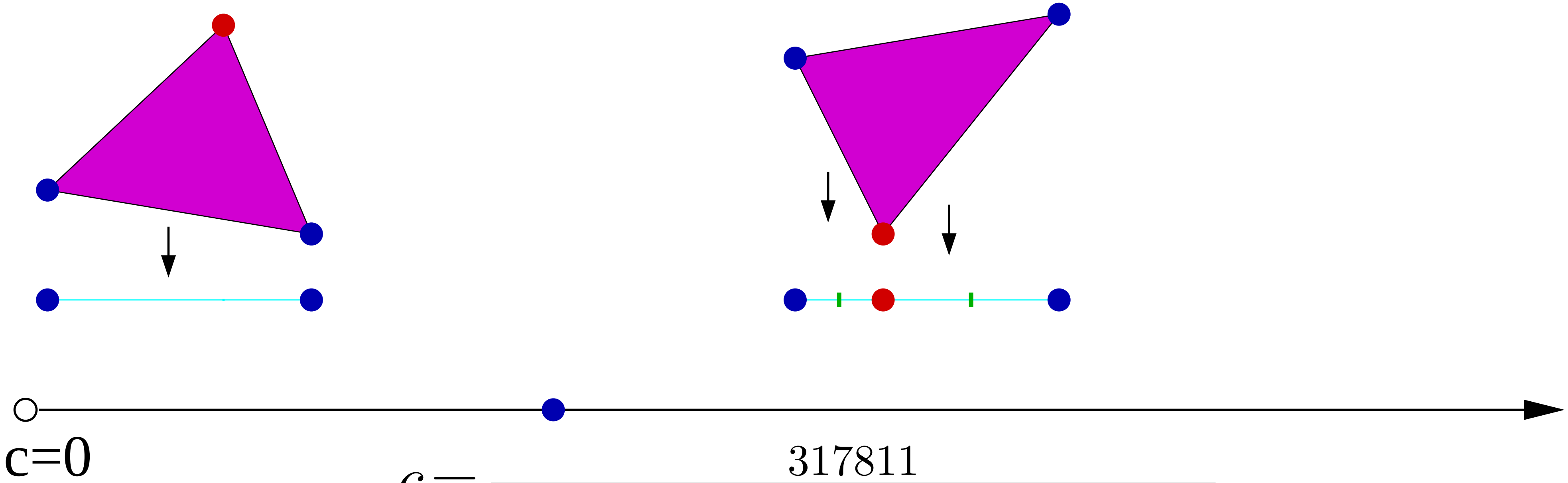
$f = f' = 0$ has a root $\zeta \in \mathbb{C} \setminus \{0\} \Longleftrightarrow$
$\Delta_{\{0,d,D\}}(f) := \left(\frac{1}{D-d}\right)^{D-d} \left(\frac{-c}{-D}\right)^{-D} \left(\frac{1}{d}\right)^{d} - 1$ vanishes!

$b^2 - 4ac \longrightarrow \mathcal{A}$-DISCRIMINANTS

We call any connected component of the **complement** of

$$\{c \in \mathbb{R} \setminus \{0\} \mid \bar{\Delta}_{\{0,d,D\}}(c) = 0\}$$

a **(reduced) discriminant chamber**.



$$c = \frac{317811}{196418^{196418/317811}\,121393^{121393/317811}}$$

$$\approx 1.944526275\ldots$$

$b^2 - 4ac$ ON STEROIDS

So you can decide whether

$$1 - cx^{196418} + x^{317811}$$

has 0, 1, or 2 positive roots, just by checking whether $196418^{196418}121393^{121393}c^{317811} - 317811^{317811}$ is <0, $=0$, or >0...

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...and, availing to Morse Theory, this generalizes to arbitrary n...

TRIVARIATE EXAMPLE

Consider

$$c_1 + c_2 x_1^{999} + c_3 x_1^{73} x_3 + c_4 x_2^{2009} + c_5 x_1^{2009} x_2^{6027} x_3^{18081} \ldots$$

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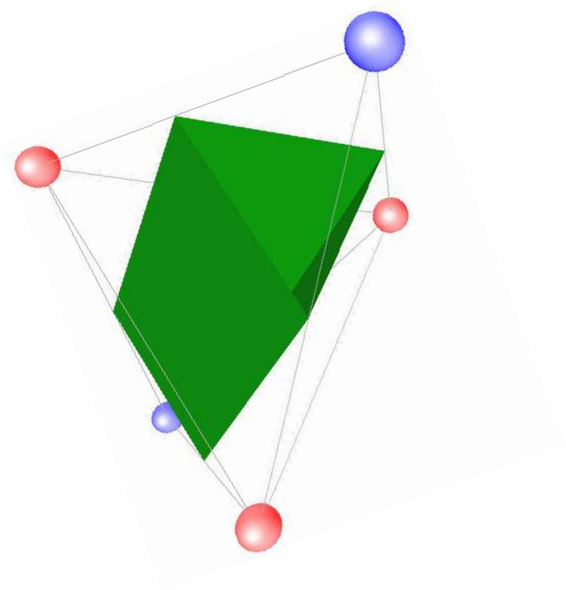
being positive...



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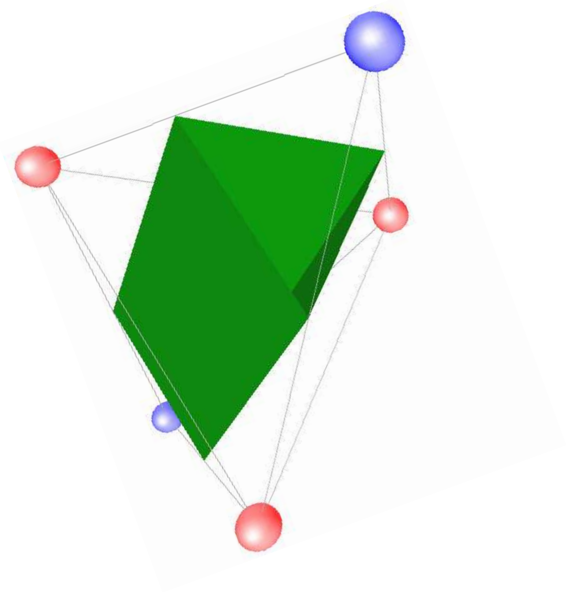
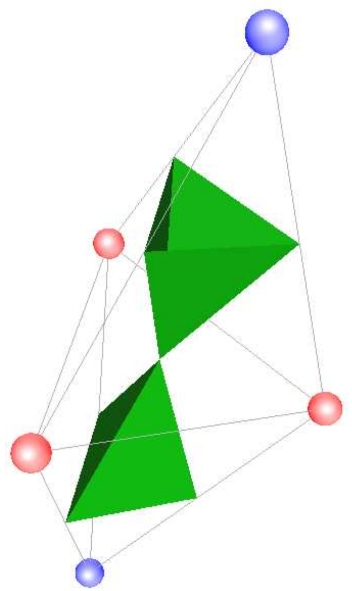
being positive... zero...





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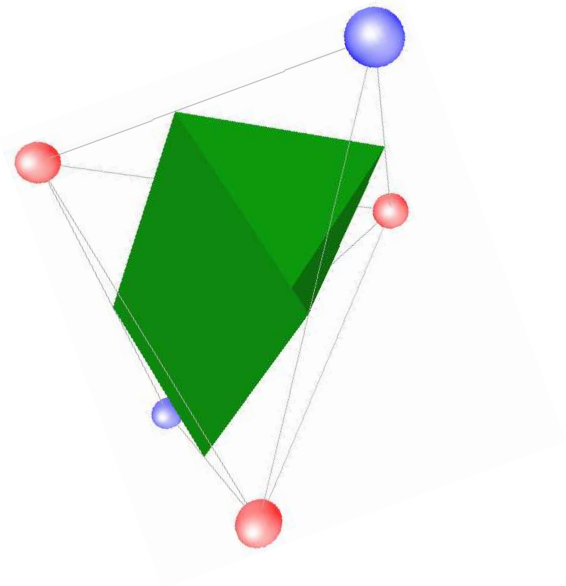
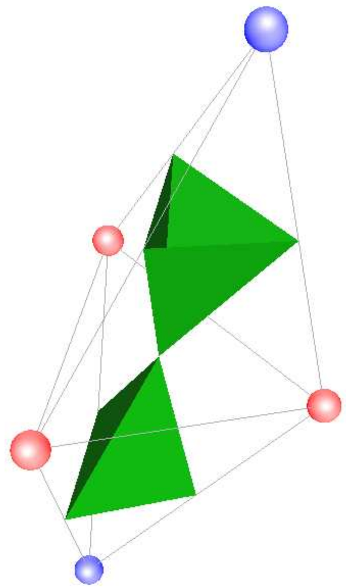
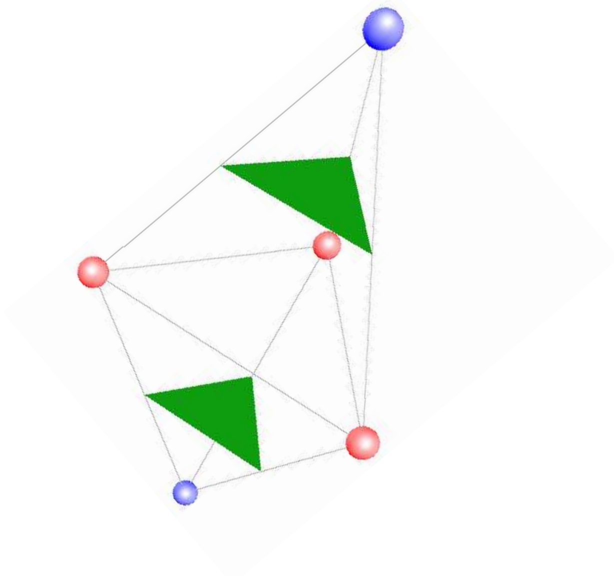
being positive... zero... or negative.







THE ALGORITHM

Given $f(x) := \sum_{i=1}^{n+2} c_i x^{a_i}$ with $\mathcal{A} := \{a_1, \ldots, a_{n+2}\}$ of cardinality $n+2$ and $\begin{bmatrix} 1 & \cdots & 1 \\ & \mathcal{A} & \end{bmatrix}$ of rank n...

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4. Check if $\Delta_{\mathcal{A}}(f) \stackrel{?}{=} 0$...

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$\boxed{5.}$ Check if $s_f \Delta_{\mathcal{A}}(f) \stackrel{?}{>} 0$...

HIGH PROBABILITY?

Corollary 1 *For uniformly distributed sign, you can decide* $Z_+(f) \stackrel{?}{=} \emptyset$ *in* $\mathbf{NC}^2$ *on a fraction of* $1 - \frac{1}{2^{n+2}}$ *of the inputs, even if* n *is not fixed a priori!*

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Other improvements?

DIOPHANTINE REFINEMENT

[Bihan-Rojas-Stella] *Fix n. Then for any "honest" n-variate $(n+2)$-nomial f, one can decide $Z_+(f) \stackrel{?}{=} \emptyset$ in $\mathbf{P}$.*

Corollary 2 *[Rojas, 2008] Assuming* **Baker's refinement of the abc-Conjecture**, *we have polynomiality in n as well!*

DIOPHANTINE REFINEMENT

Corollary 2 *[Rojas, 2008] Assume* **Baker's refinement of the abc-Conjecture**. *Then for any n and any "honest" n-variate $(n+2)$-nomial f, one can decide $Z_+(f) \stackrel{?}{=} \emptyset$ in* $\mathbf{P}$.

Baker's Refined abc-Conjecture (1998)

Let $N(s) := \prod\limits_{p|s \text{ and } p \text{ prime}} p$ *for any integer* s...

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Let $N(s) := \prod_{p|s \text{ and } p \text{ prime}} p$ *for any integer* s *and define*

$\omega(s) := \#\{p \ : \ p|s \text{ and } p \text{ prime}\} \ldots$

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$$c = O\left(\frac{\log^{\omega(abc)} N(abc)}{\omega(abc)!} N(abc)\right).$$

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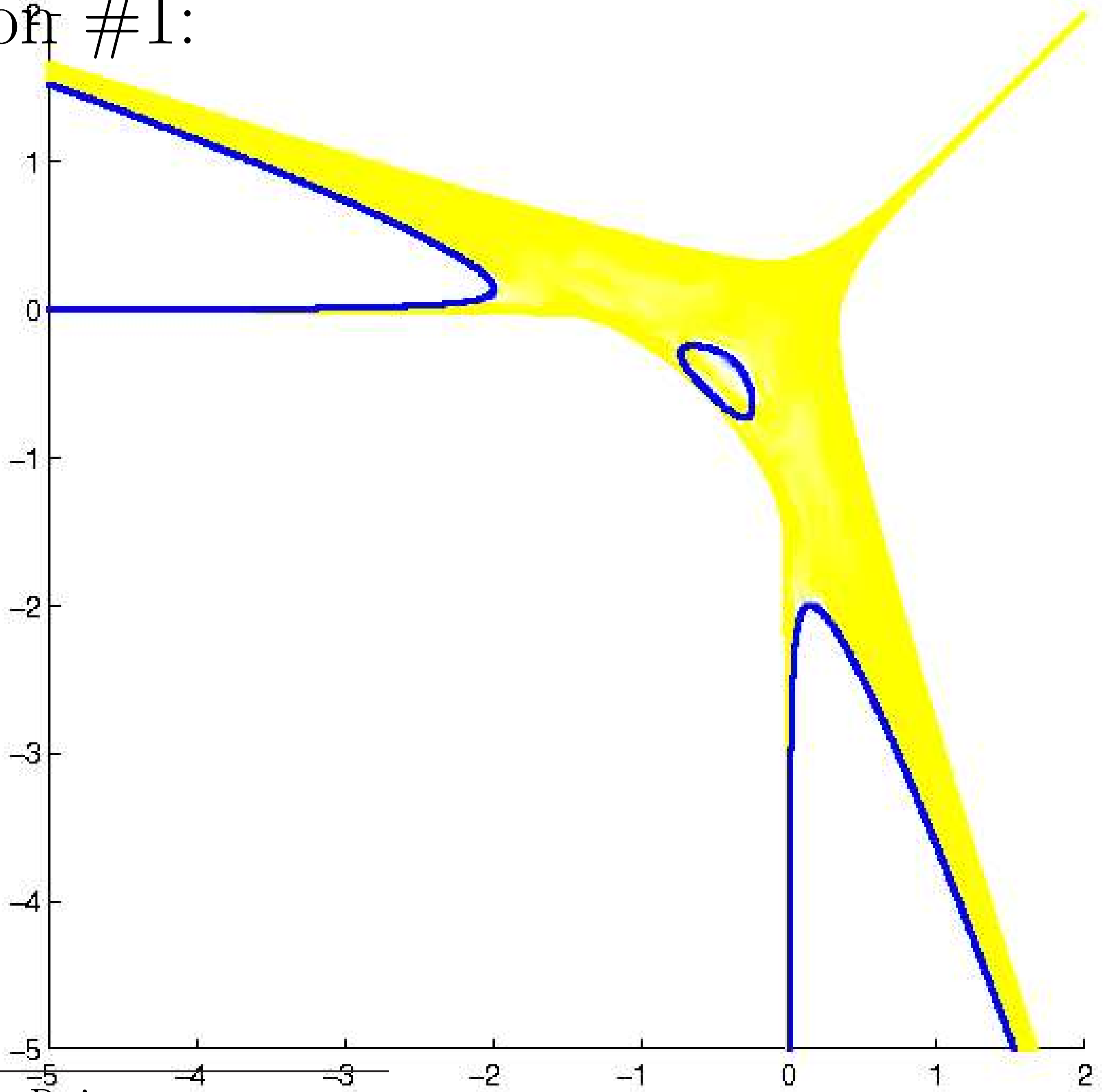
$$c = O\left(\frac{\log^{\omega(abc)} N(abc)}{\omega(abc)!} N(abc)\right).$$

Note: Baker's Refined abc-Conjecture implies: (1) Effective Falting's Theorem [Elkies '91], (2) Effective Roth's Theorem [Bombieri '94, Surroca '07], (3) non-existence of Siegel zeroes for certain L-functions [Granville '00]. Conversely, suitably sharp versions of (1) or (2) imply variants of abc! [Surroca '07, van Frankenhuysen '07]

n-VARIATE $(n+3)$-NOMIALS?

Obstruction #1:



Thank you for listening!

Please see...

`www.math.tamu.edu/~rojas`

for on-line papers and further information.