A Gröbner-Free Alternative to Solving and a Geometric Analogue to Cook's thesis

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MATHEMATICS:

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Isaac Asimov, Second Foundation.

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- Given $\ell(X_1, \ldots, X_n)$ a non zero linear form
- Find a non zero univariate polynomial p and s divisors $g_1, \ldots, g_s$ in $\mathbf{k}[X_1, \ldots, X_n]$ such that the following division holds:

$$p \circ \ell = f_1 g_1 + \cdots + f_s g_s.$$

PoSSo cont.

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- Elimination of the variables $X_1, \ldots, X_n$ among the equations $l(X_1, \ldots, X_n) - Y, f_1, \ldots, f_s$

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- In our minds the problem is solved (at least theoretically ... ;) numerically or symbolically, by recombination of the univariate solvings.
- From a practical point of view, this is another story. How to avoid this recombination step?

PoSSo cont.

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- This condition is generic in Thom's sense.
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The variety V is parametrized polynomially by the zeroes of p. This representation goes back at least to Kronecker [82]. It was rediscovered in computer algebra again and again under the names Shape Lemma or Rational Univariate Representation.

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- V is of dimension zero if s equals n.

Geometric Resolution (static)

- Geometric version

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  - This extension can be realized by a primitive element.

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$$q(x_1,\ldots,x_{n-i},T)=0,\quad \begin{cases} \frac{\partial q}{\partial T}x_{n-i+1} & = & w_{n-i+1}(x_1,\ldots,x_{n-i},T), \\ & \vdots & \\ \frac{\partial q}{\partial T}x_n & = & w_n(x_1,\ldots,x_{n-i},T), \end{cases}$$

where

- $q, w_{n-i+1;},\ldots,w_n$ are polynomials in $\mathbb{Q}[x_1,\ldots,x_{n-i},T]$
- q is monic in T and square free.

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  Aldaz, Bank, Beltran, Bostan, Cafure, Castaño, Castro, Colin D'Alfonso, Durvye, Fitchas, G, Grimson, Hägele, Heintz, Jeronimo, Krick, Lecerf, Lehmann, Llovet, Mandel, Marchand, Massaccesi, Matera, M'bakop, Montaña, Morais, Morgenstern, Ollivier, Pardo, Puddu, Sabia, Salvy, Schost, Sedoglavic, Smietanski, Solernó, Turull, Wachenchauzer, Waissbein, . . .

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- Height of a polynomial

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Extension to the representation of integers [Hägele–Montaña].

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- For SLP's we speak of length, i.e. the number of arithmetic operations
- Each arithmetic processor can be become a boolean circuit manipulating bits. But we have to take into account the of the height of the scalars involved, to define in the same way the complexity.

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Input Complexity: the 4-uple (n, d, h, L)

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  - (Ordinary) Bézout Theorem: $\delta \leq d^n$
  - (Arithmetical) Bézout Theorem: $\eta \leq hd^{An}$, A universal constant
  - Degree and height can be arbitrarily smaller.

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This is a *theoretical* theorem. Anyway we refer from now on to evaluation techniques if SLP encoding is used.

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  Equidimensional decomposition, irreducible decomposition modulo factorization
- Primary decomposition of an arbitrary variety of dimension 0 outside an hypersurface.
  Multiplicities and even more: description of the local algebras.

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– Software `KRONECKER`

– Package written in `MAGMA` by G. Lecerf, improvements by Schost and Lehmann

– Home page of Lecerf:
`http://www.math.uvsq.fr/`

Miscelleanous

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– Ritt's resolvent of a prime differential ring [Solerno et al.]

Lower Bounds and Optimality Considerations

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Folklore: Elimination and Cook's question are related. Here is a precise statement due to [Heintz–Morgenstern 1993]:

Theorem *If $\mathcal{P} \neq \mathcal{NP}$ holds, there exists no uniform well parallelizable algorithm such that:*

- *the input $f_1, \ldots, f_s \in \mathbf{k}[X_1, \ldots, X_n]$ is given by a well parallelizable division–free straight–line program of length L*
- *it solves the affine zero dimensional problem in sequential time $(Ld)^{O(1)}$*
- *the output polynomial $Q(Y)$ belonging to $(f_1, \ldots, f_s)$ is of degree $\deg V$ and is given by a well parallelizable division–free straight–line program of length $(Ld)^{O(1)}$.*

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Example

Conditions:

- $f_1, \ldots, f_n$ reduced regular sequence
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yield a constructible $W \subset \mathbb{C}^m$ with $m = (n+1)\binom{d+n}{n}$.

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- χ_q its characteristic polynomial.

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- Image of $q : V(f_1, \ldots, f_n) \longrightarrow \mathbb{C}$?
- $\prod_{x \in V}(y - q(x))$
- Multiplication by q induces an endomorphism $\mathbb{C}[X_1, \ldots, X_n]$
- χ_q its characteristic polynomial.

Questions?

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Nullstellensatz

- $V(f_1, \ldots, f_n, q) = \emptyset$?
- q vanishes on some point of V?
- Constant term of χ_q is 0?
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Ideal Membership

- $q \in (f_1, \ldots, f_n)$?
- q vanishes on V?
- All non leading terms χ_q are 0?
- $Id : \mathbb{C}^D \longrightarrow \mathbb{C}^D$ zero or not?

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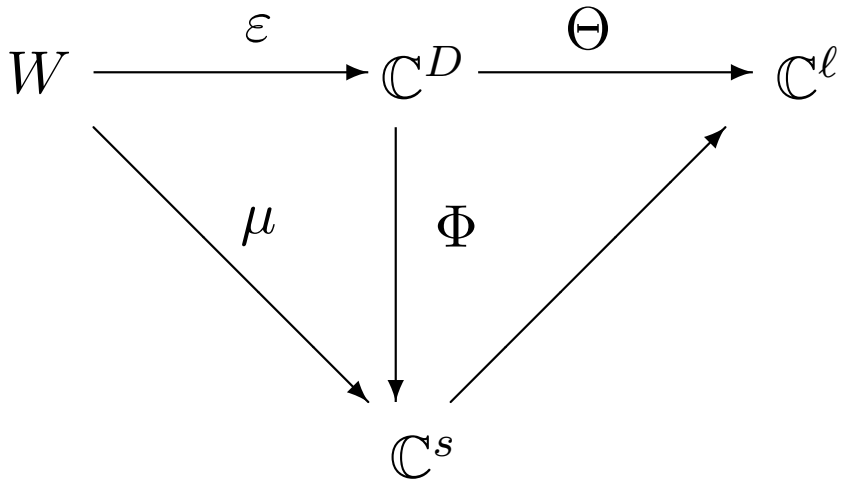
From now on we will be in this situation, and D will be d^n.

Alternative: Change the Data Structure



Dense representation : explicit writing of ε too large. Try:

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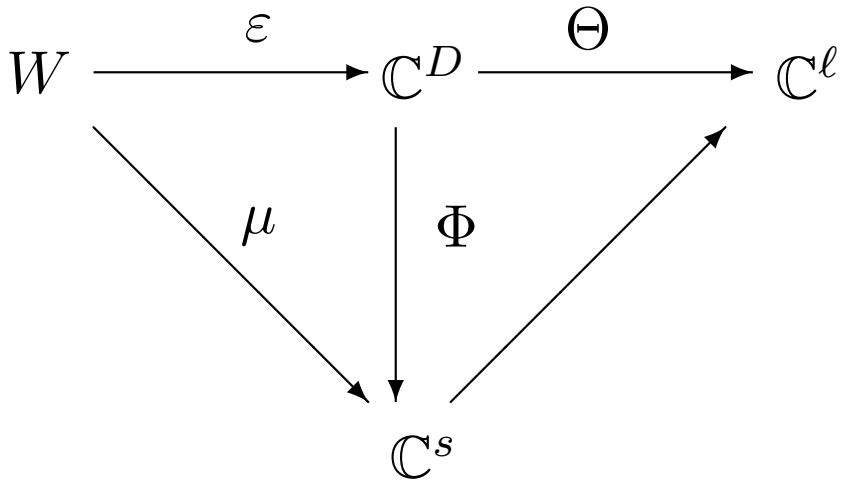

Dense representation : explicit writing of ε too large. Try:

- More efficient data structure?

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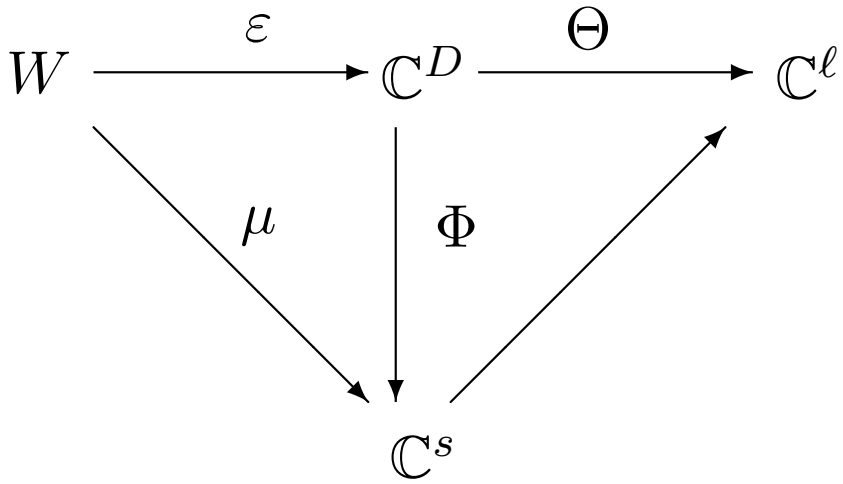

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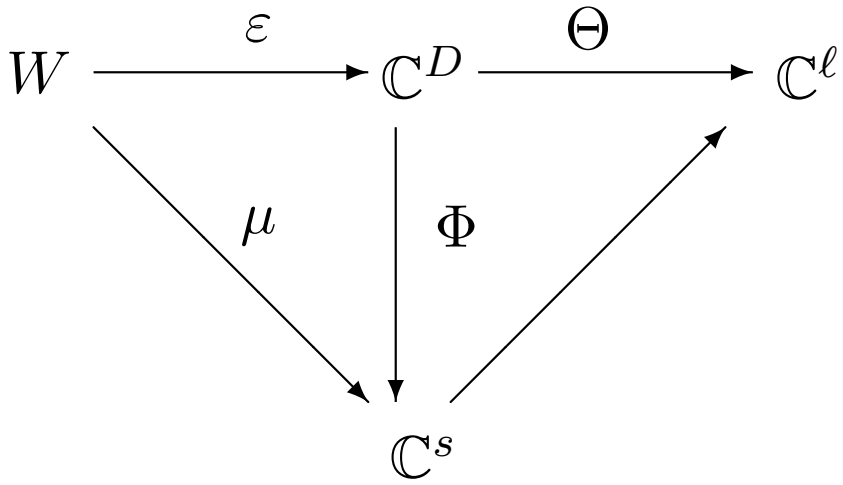

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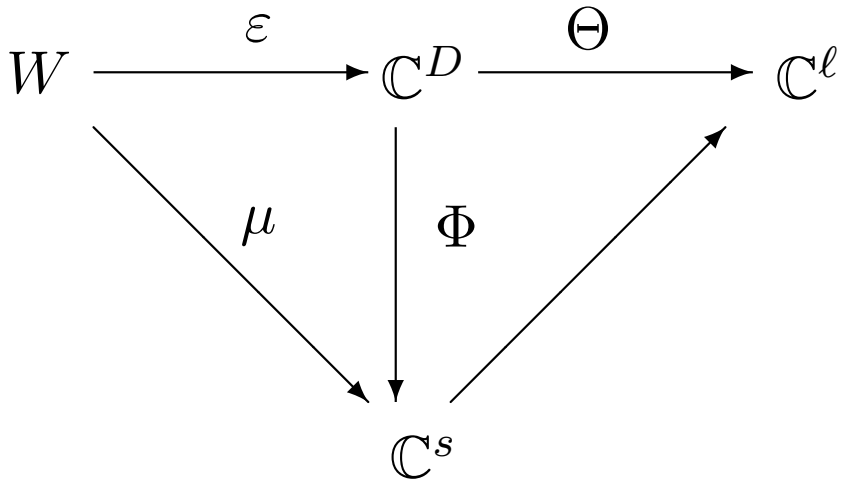

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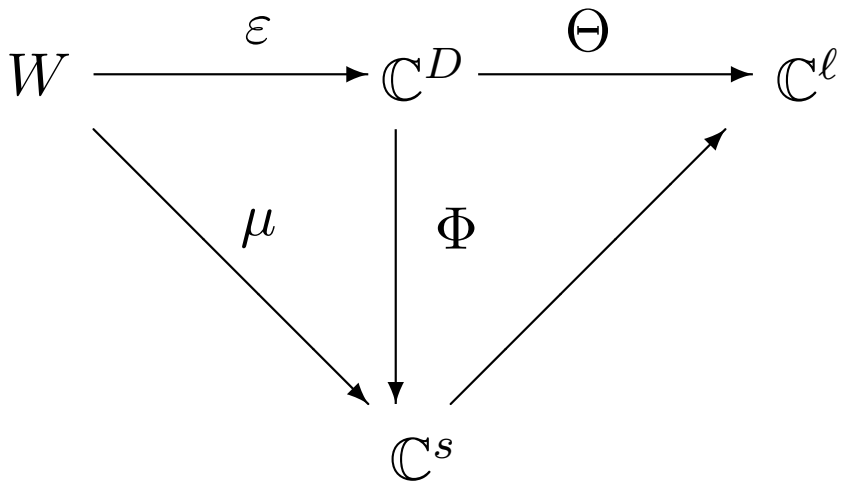

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- Interpolation map $\mathrm{Im}\ \Phi \longrightarrow \mathrm{Im}\ \varepsilon$
- Black–box $\mu = \Phi \circ \varepsilon$

Correct Test Sequences [Heintz–Schnorr 82]

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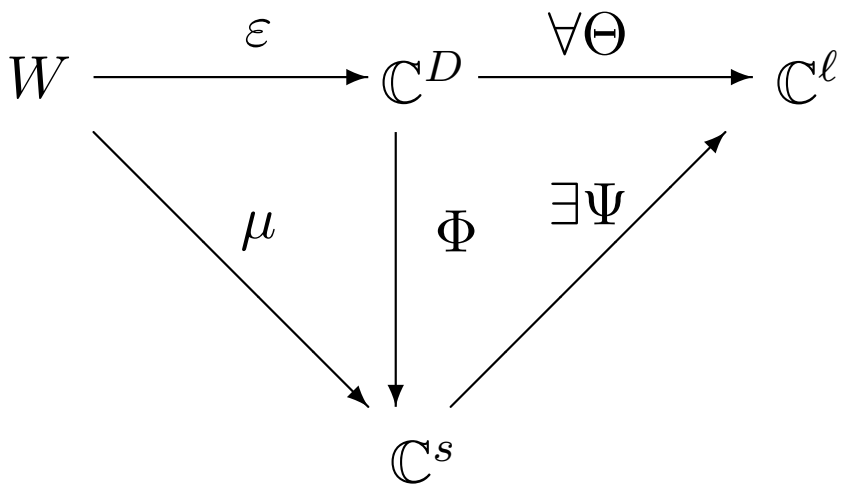
Question answered: when $\varepsilon(w) = 0$?

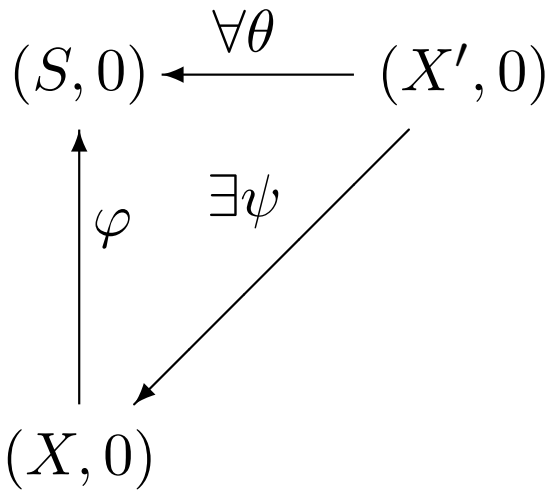
- There are elimination procedures, suitably encoded, whose output length is linear in the input length
- No exponential time lower bound for Elimination Theory can be found only by means of output length

Universal Elimination Procedures

Their output, suitably encoded, contains information enough to answer ANY question you may ask in the future on any instance of W.



Convincing the pure mathematician



- Distinguished point 0
- $\chi_g = T^D$
- Thom's organizing center: worst singularity!
- This corresponds to the worst problem, i.e. the ideal membership!!
- The dimension of the basis of the (almost) universal unfolding of T^D is $D-1$, hence the length of the output encoding s is exponential in the input length.

Conclusion: Universal Elimination Requires Exponential Time

- Efficient Elimination Methods must look for either partial answers discarding some questions
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- Or accept to run in exponential time!

Even no further hope in direction of numerical analysis data structures.

Convincing the computer scientist

Any elimination procedure which is:

- parametric
- parsimonious with respect to *branchings* and *divisions*
- must necessarily have a non–polynomial sequential time complexity
- even if highly efficient data structures (as e.g. the arithmetic circuit encoding of polynomials) are used.

Convincing the numerical analyst: the hardness of Polynomial System Solving

- Let be given an arbitrary continuous data structure encoding systems of polynomials
- together with this data structure a universal elimination algorithm, say $\mathcal{P}$,
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- Then $\mathcal{P}$ cannot be a polynomial time algorithm.

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Motivation: Wavelet construction via algorithmic real algebraic geometry

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If the system of those equations has real solutions at all, the solution set can consist of a finite number of points or can be a variety of positive dimension.

In the literature published on this topic, only example problems with a finite solution set were presented. For the computation of those examples it was sufficient to solve quadratic equations in one or two variables. This is easily done with the help of the tools of common computer algebra systems.

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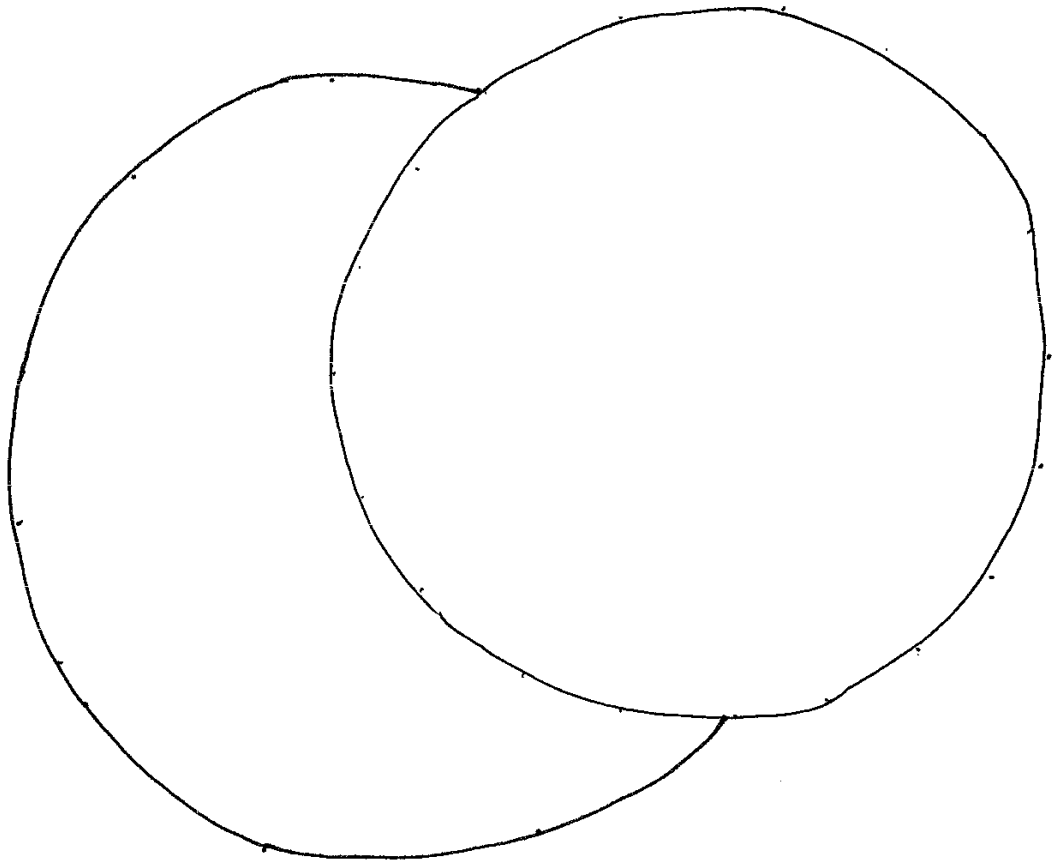
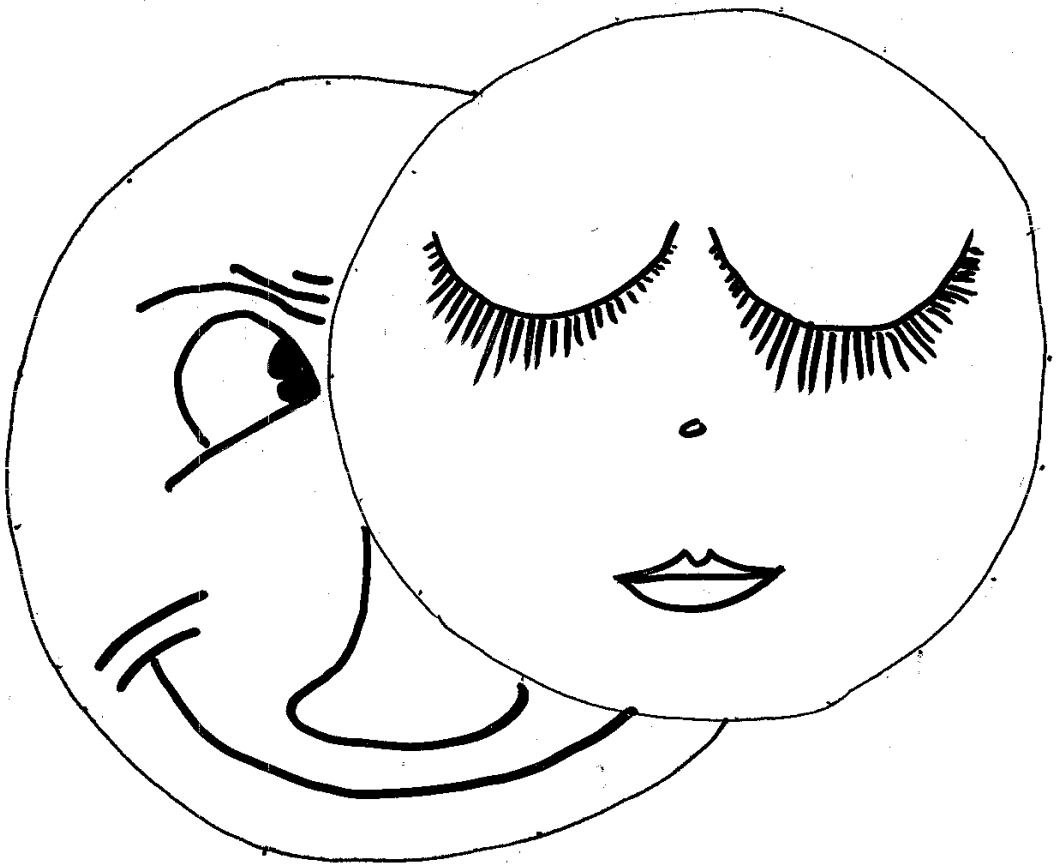
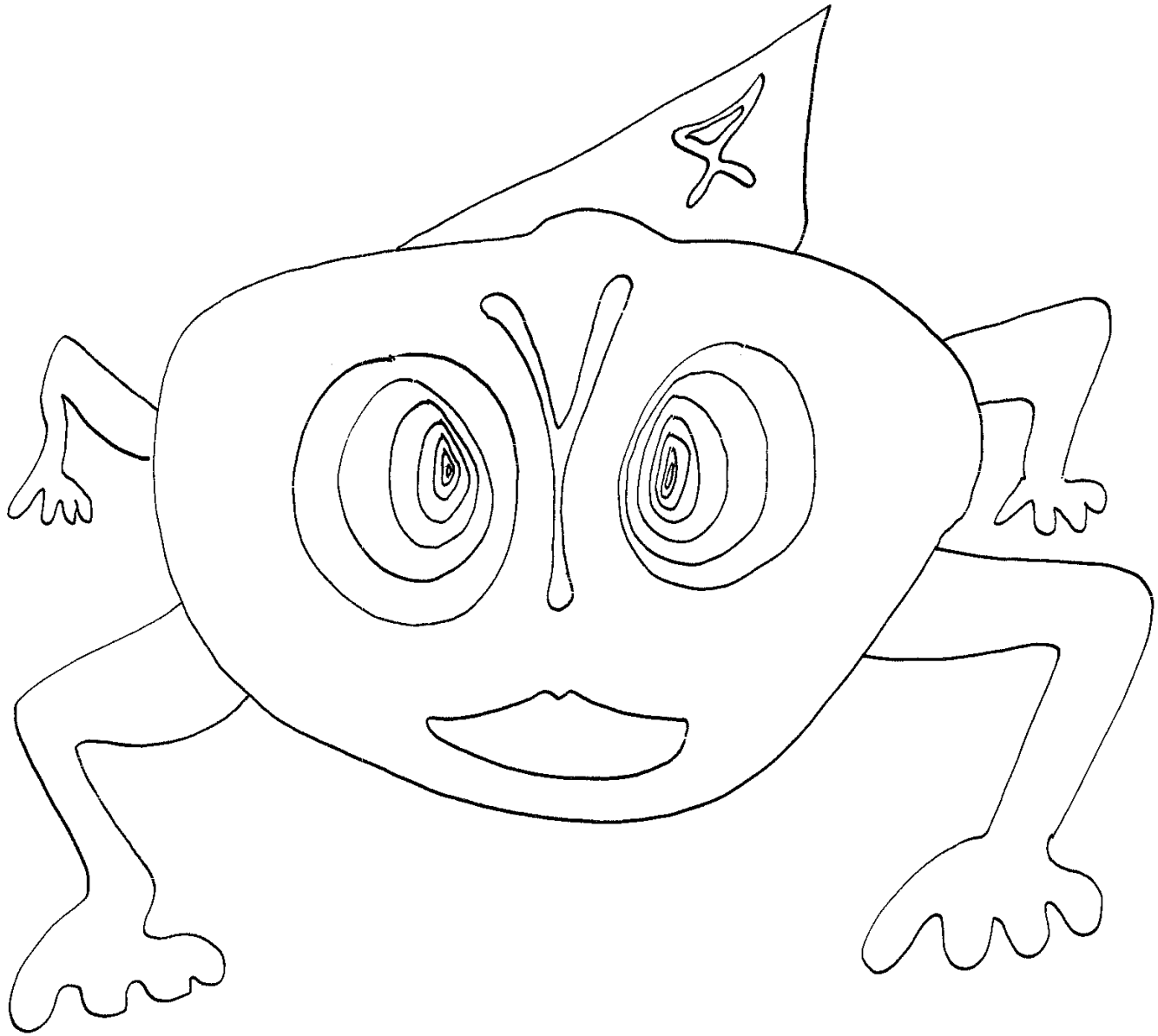
To characterize the set of real solutions of a system of polynomial equations it is a first step to find at least one point in each connected component.

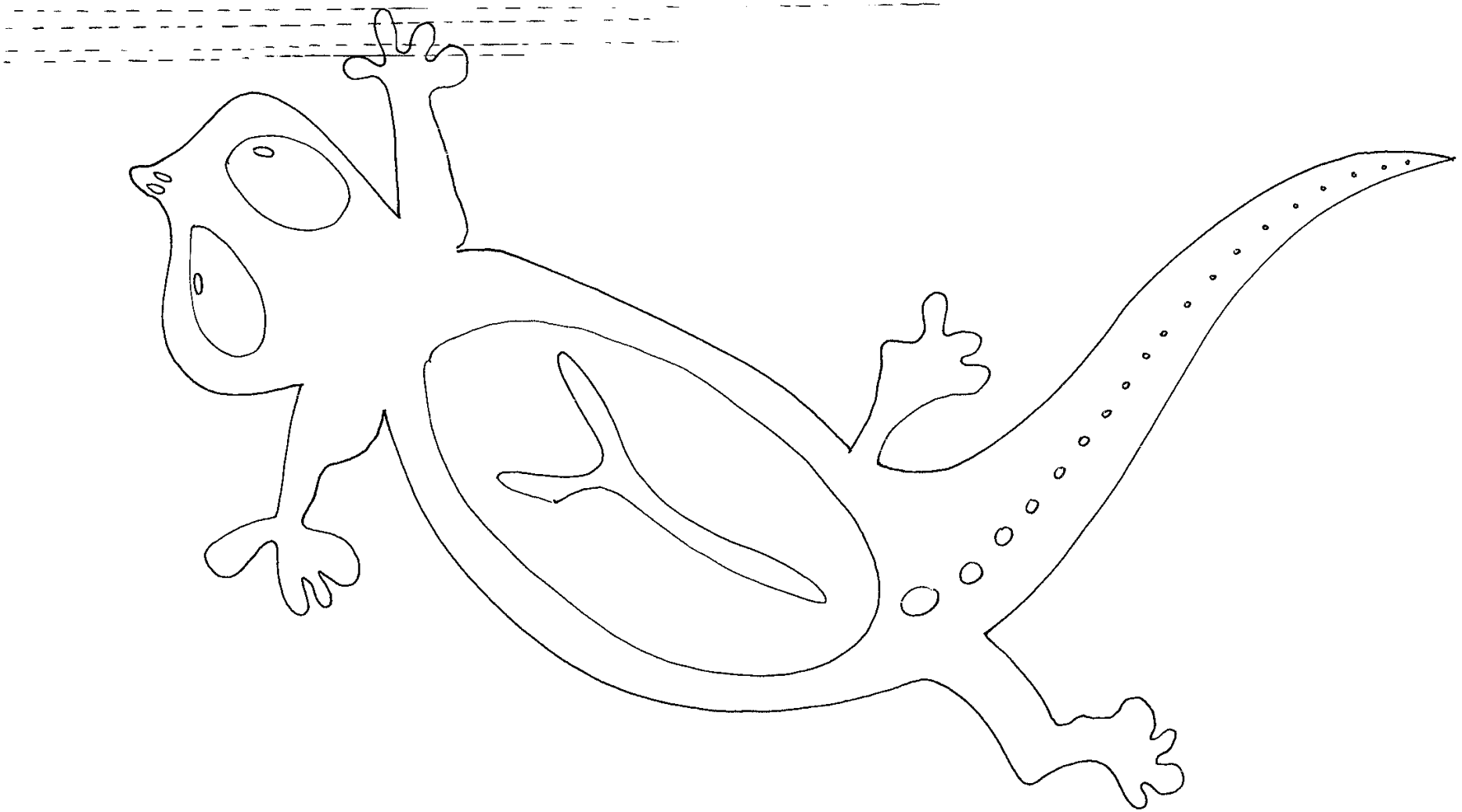
It turns out that the algorithm of the TERA–project performes very well with this task and is able to solve a larger number of examples than the best known commercial polynomial solvers.

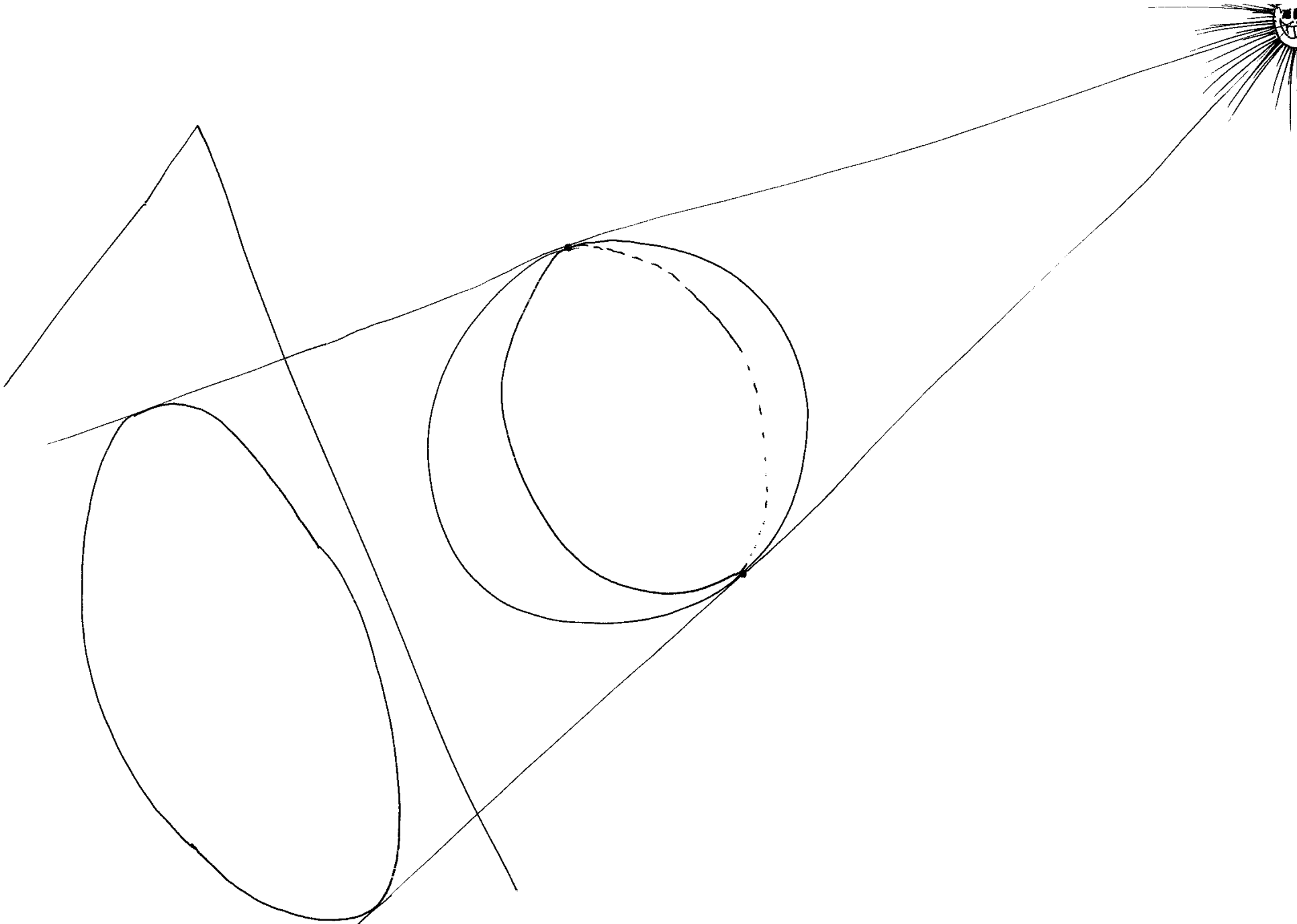






An algebraic curve





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- In particular to find efficiently one representative point in every connected component of a real algebraic variety constitutes a fundamental problem in real algebraic geometry. We tackle this problem by exploiting the

  geometric solving of generic or (sufficiently generic) polar varieties (by the algorithm `Kronecker`)

- Several generalisations of the notion of polar variety allows us to drop successively the assumptions of

  hypersurface, compactness and smoothness.

Polar varieties: Notations

Let A be a **generic** $((n-1)\times n)$–matrix.

For an index i, fixed between 1 and $n-1$, we denote by

$$A_i := \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n-i,1} & a_{n-i,2} & \cdots & a_{n-i,n} \end{bmatrix},$$

the sub–matrix of A formed by the lines $1, 2, \ldots, n-i$ of A.

- $f \in \mathbb{Q}[x_1, \ldots, x_n]$
- $\mathcal{V} := \mathcal{V}_{\mathbb{C}}(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\}$
- $\mathcal{V}_{\mathbb{R}}$ its real trace $\mathcal{V}_{\mathbb{C}} \cap {\mathbb{A}_{\mathbb{R}}}^n$

Polar varieties: Definitions

The i–th (open) polar variety of $\mathcal{V}_{\mathbb{R}}$

is defined by

$$\Delta_i(A) := \{x \in \mathcal{V}_{\mathbb{R}} \setminus Sing\ \mathcal{V}_{\mathbb{R}} \mid T_x\ \mathcal{V}_{\mathbb{R}} \text{ not transversal to } \ker A_i\}.$$

$\Delta_i(A)$ is the set of all points in $\mathcal{V}_{\mathbb{R}}$,
where the $((1+n-i)\times n)$–matrix

$$\mathcal{A}_i(x) := \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ & A_i & \end{bmatrix} \text{ is not of maximal rank, i.e.,}$$

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The i–th (closed) polar variety of $\mathcal{V}_{\mathbb{R}}$

is defined as the Zariski–closure of the i–th open polar variety.

Polar varieties: Determinantal Equations in the Smooth Case

- If the hypersurface $\mathcal{V}_{\mathbb{R}}$ is smooth, one obtains natural equations by adjoining to $f = 0$ all maximal minors of the matrix $\mathcal{A}_i(x)$.
- According to the maximal codimension of determinantal varieties, the expected codimension of the i–th polar variety is i,

$$n - (1 + n - i) + 1 = i \ .$$

- Corresponding to the flag

$$\ker A_{n-1} \subset \ldots \subset \ker A_1$$

the polar varieties form a decreasing sequence of sub–varieties of $\mathcal{V}$ with expected codimension $1, \ldots, n-1$.

Notations and problem

- $f \in \mathbb{Q}[x_1, \ldots, x_n]$
- $\mathcal{V} := \mathcal{V}_{\mathbb{C}}(f) := \{x \in \mathbb{C}^n \mid f(x) = 0\}$
- $\mathcal{V}_{\mathbb{R}}$ its real trace $\mathcal{V}_{\mathbb{C}} \cap {\mathbb{A}^n}_{\mathbb{R}}$
- The equation of the hypersurface $f = 0$ is assumed to be regular, i.e. : $\mathcal{V}_{\mathbb{R}}$ is non empty and $grad(f)$ does not vanish on any of its connected component.
- Degree of $f \leq d$

FIND AT LEAST A POINT IN EVERY CONNECTED COMPONENT OF $\mathcal{V}_{\mathbb{R}}$

Small history of the problem

- Grigoriev, Grigoriev/Vorobjov '87 , '88
- Complexity $d^{O(n)}$:
  Heintz/Roy/Solerno '89, '90
  Barvinok '91
  Renegar '92, ,'98
  Canny/Emiris, Canny '95, '98
  Blum/Cucker/Shub/Smale '97
  Cucker/Smale '98
  Basu/Pollack/Roy '95, '98
- More recent papers:
  Rouillier/Roy/Safey el Din
  Aubry/Rouillier/Safey el Din
  Safey el Din/Schost, Safey el Din
  ...
- Our contributions '97, '98, '00, '02, '05, '07, '08

Polar Varieties: Quantification of Data Complexity

- $f \in \mathbb{Q}[x_1, \ldots x_n]$ regular equation of a real smooth hypersurface

$$\mathcal{V}_{\mathbb{R}} = f^{-1}(0) \subset \mathbb{R}^n, \quad \mathcal{V}_{\mathbb{R}} \text{ compact or not}$$

- Degree of $f \leq d$
- Evaluation complexity of $f \leq L$
- Maximal **degree of the complex polar varieties** δ;
  (always holds $\delta \leq d^n$, $\quad d^n$ is the **Bézout** number)
- If $\mathcal{V}_{\mathbb{R}}$ is not compact we use **generalized** polar varieties (see later).

The smooth past: find a point in every connected component of $\mathcal{V}_{\mathbb{R}}$ when smooth

: Complexity Theorem

- **Intrinsic Complexity** (number of arithmetic operations in $\mathbb{Q}$)

$$L(nd\delta)^{O(1)}$$

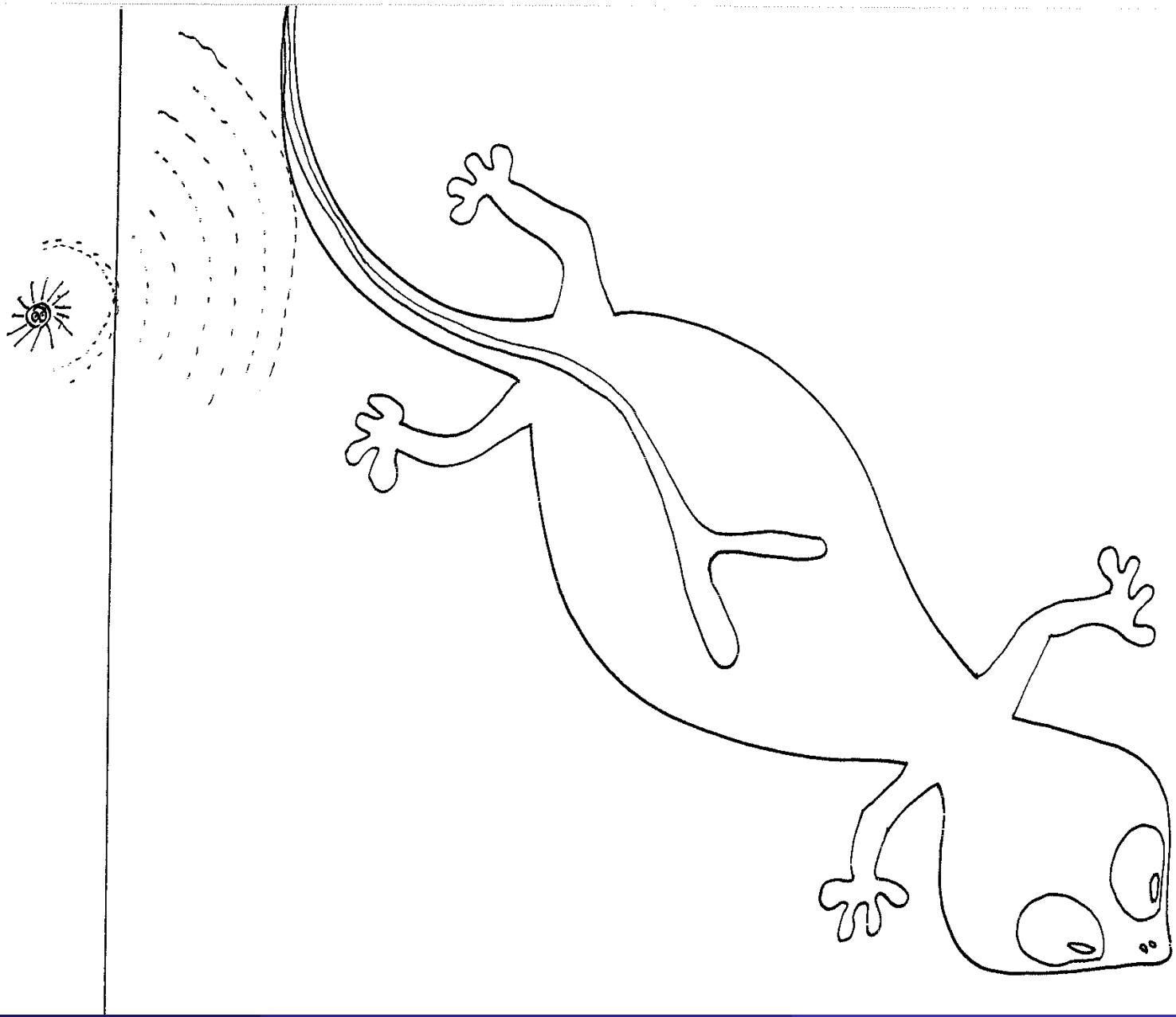
- Existence theorem
- Probabilistic version
- **Extrinsic complexity:**

Linear in $\mathbf{L}$

Polynomial in the Bézout number $\mathbf{d^n}$

We meet the best known extrinsic complexity bounds.

A non compact algebraic curve



The non compact case

We consider a larger parameter space, the product of

- a flag variety,
- a variety of hyperquadrics and
- a variety of hyperplanes.

The RADAR technology.

Again we arrive in the same complexity class!

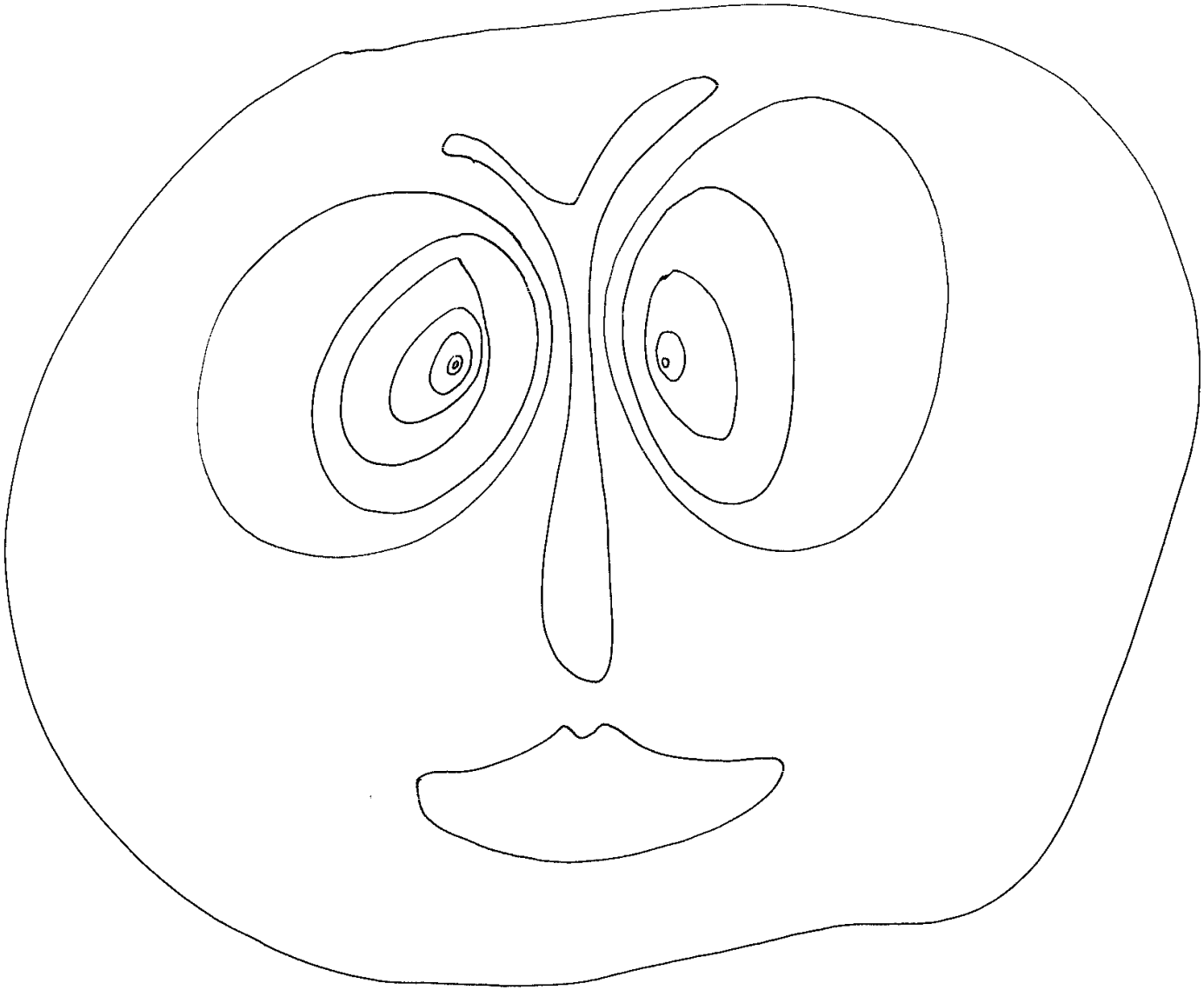
Complete intersection case; still smooth

- $f_1, \ldots, f_p$ a reduced regular sequence
- $\mathcal{V} := \{f_1 = \cdots = f_p = 0\}$

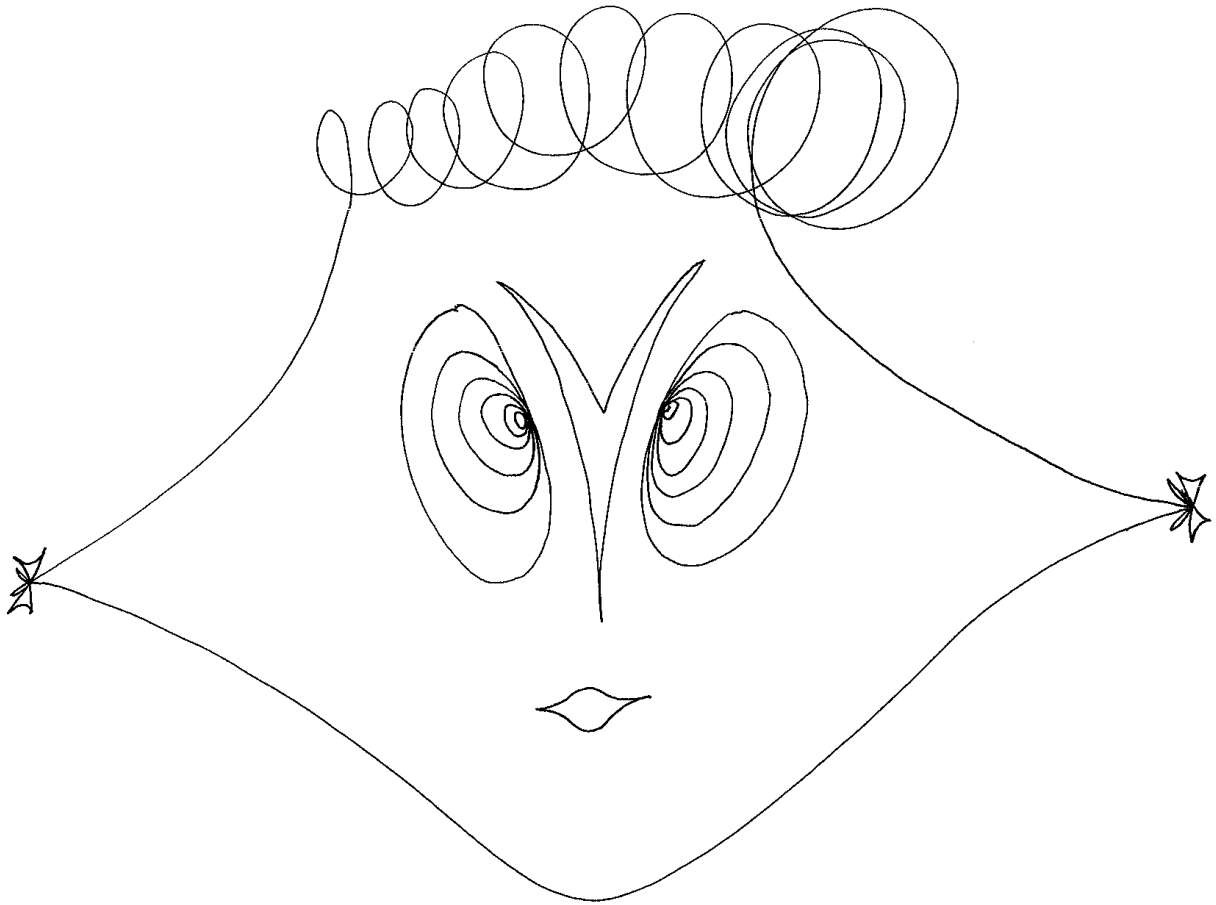
Up to a combinatorial factor we obtain the

same complexity class.

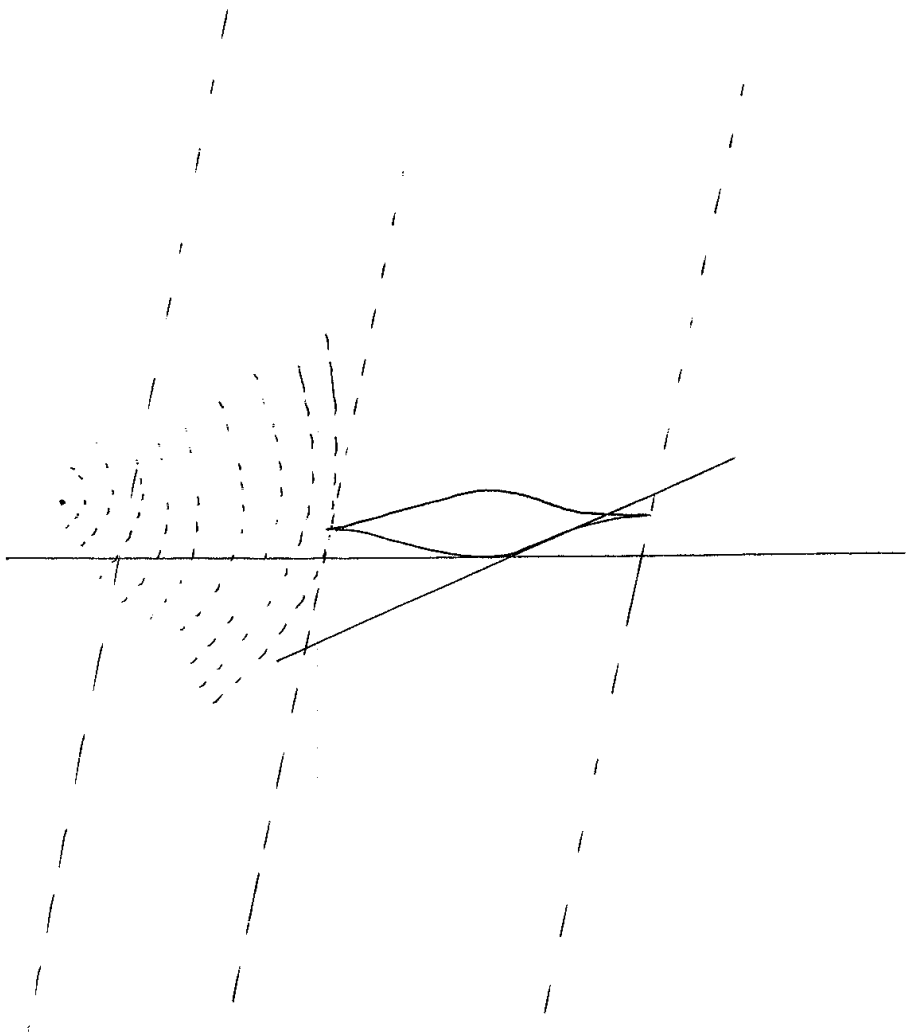
Again the old algebraic curve!



A new fresh one!!

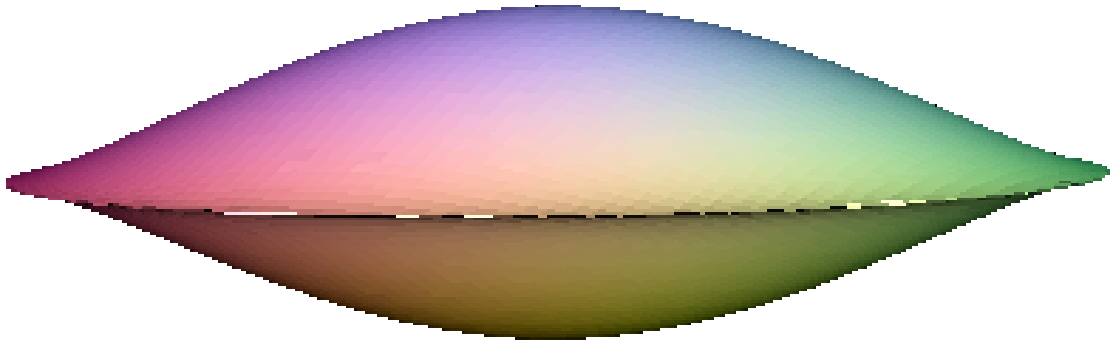


Thom's lips



The hat of the bishop of Jerusalem

$$f(x,y,z) := z^2 - \varepsilon^2(x^2+y^2-1)^3 = 0, \quad \varepsilon \quad \text{small}$$



The singular case

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We no longer have natural equations since ...

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Worse

We no longer have natural equations since . . .

. . . The determinantal description

defined above by the maximal minors is no longer valid since it contains all the singular locus.

LISSIFICATION AND STRATIFICATION VERSUS DESINGULARIZATION

Canonical desingularization of determinantal varieties – à la ROOM–KEMPF

- In the following let i be a fixed index between 1 and $n-1$. We consider the linear system

$$J(f)(x)^T\lambda + A^T\mu = 0$$

- with x in $\mathcal{V}$ and (λ, μ) in $\mathbb{A}^1 \times \mathbb{A}^{n-i}$.

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Fundamental property: x singular implies $\mu = 0$

The singular case

The parameter space E_i

$$E = E_i := \{(A, \lambda, \mu) \in \mathbb{A}^{n(n-i)} \times \mathbb{A}^1 \times \mathbb{A}^{n-i} \,|\, rang\, A_i = n - i,\ \mu \neq 0\}$$

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Group action

On E there is a natural right action of the group $G := Gl(n-i) \times Gl(1)$:
Let $g := (B, t) \in G$ and $e := (A, \lambda, \mu) \in E$

$$E \times G \to E, \qquad (e, g) \mapsto e \cdot g := (B^T \cdot A, t\lambda, tB^{-1} \cdot \mu)).$$

The singular case

The parameter space E_i

$$E = E_i := \{(A, \lambda, \mu) \in \mathbb{A}^{n(n-i)} \times \mathbb{A}^1 \times \mathbb{A}^{n-i} \,|\, rang\, A_i = n - i,\ \mu \neq 0\}$$

Group action

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Quotient space

G induces on E_i an equivalence relation $\sim$.
We denote by $E_{/\sim}$ the set of G–orbits.

The singular case: small miracle

- The categorical quotient $E_i/\sim$ is a geometric quotient, i.e., is endowed with an algebraic variety structure – G is a " good " group (linearly reductive).

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- The categorical quotient $E_i/\sim$ is a geometric quotient, i.e., is endowed with an algebraic variety structure – G is a " good " group (linearly reductive).
- This quotient is closed!
- Moreover it is a differentiable manifold.The points in a typical chart are of the form

$$A = \left[I_{n-i}\tilde{A}\right], \quad \mu = (1, \tilde{\mu})$$

where I_{n-i} is the identity matrix, $\tilde{A}$ a $(i \times (n-i))$–matrix, and $\tilde{\mu} := (\mu_2, \ldots, \mu_{n-i})$.

The singular case: medium miracle

- Let us consider for a fixed point x in $\mathcal{V}$ the subset defined by

$$\begin{array}{l} \dfrac{\partial f}{\partial x_1}(x)\lambda + \displaystyle\sum_{k=1}^{n-i} a_{k,1}\,\mu_k = 0, \\ \qquad\qquad \vdots \\ \dfrac{\partial f}{\partial x_n}(x)\lambda + \displaystyle\sum_{k=1}^{n-i} a_{k,n}\,\mu_k = 0, \end{array} \tag{1}$$

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- It is invariant under G.
- Therefore, we can consider the incident variety S_i of $\mathbb{A}^{n(n-i)} \times \mathbb{A}^n \times E_{i/\sim}$ defined by the same system (1) and of course the equation $f = 0$. It is closed!

The singular case: great miracle

The bipolar varieties

- The canonical projection

$$\mathbb{A}^n \times E_{i/\sim} \longrightarrow \mathbb{A}^n$$

maps S_i surjectively onto the open set of smoothness $\mathcal{V} \setminus Sing\mathcal{V}$ of smooth points (recall: $\mathcal{V} := \{x \in \mathbb{A}^n \,|f(x) = 0\}$).

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- S_i is a smooth complete intersection variety.
- S_i is **not** compact as soon as $Sing\mathcal{V} \neq \emptyset$.
- We can apply the technology developped previously for this case and consider of course its polar varieties (*generalized*). If sufficiently generic they form the bipolar varieties of de $\mathcal{V}$.
- The fibers $S_i(A)$ are (closed), generically smooth subvarieties dominating surjectively the (open) polar variety $\Delta_i(A)$.

Smoothness of the incident variety

On a typical chart S_i is defined by:

$$f(x) = 0,$$

$$\lambda \frac{\partial f}{\partial x_1}(x) + 1 = 0,$$

$$\lambda \frac{\partial f}{\partial x_l}(x) + \mu_l = 0, \qquad 2 \leq l \leq n - i$$

$$\lambda \frac{\partial f}{\partial x_l}(x) + a_{1,l} + \sum_{k=2}^{n-i} \mu_k \, \tilde{a}_{k,l} = 0, \qquad n - i + 1 \leq l \leq n.$$

At a non singular point of $\mathcal{V}$, analysing the corresponding jacobian matrix leads to:

- It is of full rank
- We can describe bipolar varieties with as many equations as its codimension (i.e. complete intersection)
- Applying the (weak) Thom's Transversality Theorem yields the generic smoothness of the fibers.

The bipolar lattice

- The bipolar varieties are organized by decreasing codimension in strictly ascending dimension:

$$B_{i,D_i} \subset \cdots \subset B_{i,j} \subset \cdots B_{i,1} \subset B_{i,0} = S_i$$

- If we let run also the index i from $n-1$ to 1, we obtain a tow–dimensional lattice of bipolar varieties.
- A walk in this lattice is a path of length at most n, starting from a zero–dimensional bipolar variety $B_{i_1,D_{i_1}}$ to end up at an orbit variety $B_{i_2,0} = S_{i_2}$
- At each step either the index i or the codimension j decreases.
- The bipolar varieties found under way along the walk, modulo suitable sections and indentifications, form an ascending sequence along which the dimension increases exactly by one. It is important to observe that their real traces are non empty, hence dense.

Algorithmic strategies

- Reversing a walk yields us an algorithmic strategy, which as soon as it finds regular points on bipolars hastens to project them on smooth real points of V
- As a bonus we obtain suitable choices of matrices A favourables to our aims
- There exists a particular walk boiling down to the previously known algorithms treating the smooth case.

The complexity theorem

Theorem.
Let $f(x_1, \ldots, x_n)$ a polynomial of degree $d \geq 2$ defining as above complex and real hypersurfaces V and $V_{\mathbb{R}}$. Suppose that f is given by a straight–line program of size L.

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Its sequential complexity is linear in L and polynomial in d, n and an appropriate geometric quantity $\delta_{\mathcal{W}}$.
This quantity is the maximal degree of the (complex) bipolar varieties of V found along the walk. It is an intrinsic invariant of V and $\mathcal{W}$, which bounds also the number and the degree of the representative points exhibited by $\mathcal{R}_{\mathcal{W}}$.

Extrinsic Complexity for Finding One Point in Every Connected Component

It is enough to consider the last $n-1$–bipolar variety, of maximal codimension, i.e. zero-dimensional.
The system defining it satisfies:

- $d' = d$
- $n' \leq 2n$
- $\delta' \leq d^{2n}$

Conclusion:

THE SAME COMPLEXITY CLASS!

KRONECKER vs. GRÖBNER using MAGMA 2.13
Polynomial equations from the design of wavelet filters
PhD Thesis of Lutz Lehmann, advisor Bernd Bank

δ	δ^*	ktime	kmem	gtime	gmem
12	6	$1.5\,s$	$3\,MB$	$1.2\,s$	$1\,MB$
12	8	$7\,s$	$3\,MB$	$0.3\,s$	$6\,MB$
54	22	$1\,m\ 20\,s$	$7\,MB$	$3\,m\ 10s$	$38\,MB$
28	10	$16\,s$	$9\,MB$	$6.5s$	$9\,MB$
28	10	$60\,s$	$10\,MB$	$15\,s$	$10\,MB$
136	24	$30\,m$	$50\,MB$	$> 5\,h$	$> 800\,MB$
136	26	$1h\ 5\,m$	$75\,MB$	$> 5\,h$	$> 300\,MB$
32	6	$17\,s$	$7\,MB$	$2\,m\ 30\,s$	$17\,MB$
32	10	$45\,s$	$7\,MB$	$67\,s$	$21\,MB$
168	36	$1h\ 40\,m$	$98\,MB$	$> 5\,h$	$> 300\,MB$

δ: number of complex solutions
δ^*: number of real solutions
k ... : Kronecker (Lecerf, Lehmann)
g ... : Gröbner (Steel, F4 Faugère)

THANK YOU FOR YOUR ATTENTION!