Multihomogeneous resultant matrices for systems with scaled support

Angelos Mantzaflaris

GALAAD
INRIA Sophia-Antipolis – Méditerranée

joint work with **Ioannis Z. Emiris**, NKUA, Greece

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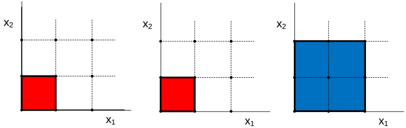
Overview

Scaled Multihomogeneous system

$$f_i(x_1, x_2, \ldots, x_n) = 0 \quad, i = 0, \ldots, n \quad \text{with } \textit{generic} \text{ coefficients}$$

Scaled case: $\boxed{\deg f_i = s_i \boldsymbol{d}}$

- Var. groups $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r) \in \mathbb{N}^r$
- Base degree $\boldsymbol{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r$
- Scalars $\boldsymbol{s} = (s_0, \ldots, s_n) \in \mathbb{N}^{n+1}$

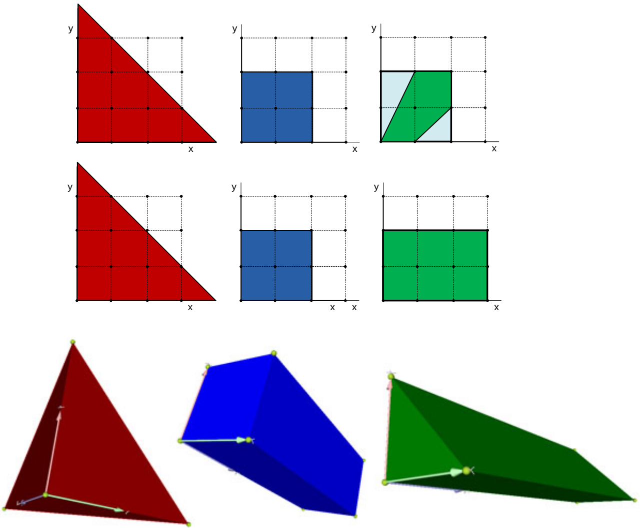


Ex.: The bilinear/biquadratic system $\boldsymbol{\ell} = (1, 1)$, $\boldsymbol{d} = (1, 1)$, $\boldsymbol{s} = (1, 1, 2)$

$$\begin{aligned}
f_0 &= a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 \\
f_1 &= b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2 \\
f_2 &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 {x_1}^2 + c_5 {x_1}^2 x_2 + \\
&\quad + c_6 {x_2}^2 + c_7 x_1 {x_2}^2 + c_8 {x_1}^2 {x_2}^2
\end{aligned}$$

affine groups: $\{x_1\}$, $\{x_2\}$, e.g. $F_0 = a_0 y_1 y_2 + a_1 x_1 y_2 + a_2 y_1 x_2 + a_3 x_1 x_2$

Resultants



Resultants

Projective resultant

$\mathcal{R}(f_0, \cdots, f_n) = 0 \quad \Longleftrightarrow \quad f_i$'s have a common (projective) root.

- $\mathcal{R}$ vanishes identically in case of infinitely many roots at infinity

Toric (or sparse) Resultant

• Define a support-specific *toric* variety X, closer to $(\mathbb{C}^*)^n$
• $\mathcal{R}(f_0, \ldots, f_n) = 0 \Longleftrightarrow \quad f_i$'s have a common root on the toric variety.

- Exploits *a priori* knowledge on the support of the equations
- Multihomogeneous resultant $\mathcal{R}_{\ell,\boldsymbol{d},\boldsymbol{s}}$:
  - an instance of the sparse resultant (set $X = \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r}$)
  - first step towards systems with arbitrary supports

Computing resultants: Matrix Formulae

- Compute a matrix M such that $\boxed{\mathcal{R} = \det M}$, or at least $\boxed{\mathcal{R} \cdot \mathcal{P} = \det M}$

Sylvester-type Matrices

Polynomial coefficients fill the non zero entries of the matrix.

reduces to:
- coeffcient matrix for linear systems
- Classic Sylvester's matrix in the univariate case
- Macaulay's matrix for homogeneous systems

Bézout-type matrices

The entries are polynomials in the input coefficients.

Hybrid matrices

There are blocks of both Sylvester and Bézout type matrices.

- Dixon's formulation

Resultants via complexes

For $\boldsymbol{m} \in \mathbb{Z}^r$ there exists the complex $K_\bullet = K_\bullet(\boldsymbol{m}; f_0, \cdots, f_n)$ of vector spaces

$$0 \to K_{n+1} \to \cdots \xrightarrow{\delta_3} K_2 \xrightarrow{\delta_2} K_1 \xrightarrow{\delta_1} K_0 \xrightarrow{\delta_0} \cdots \to K_{-n} \to 0$$

- $\delta_i \circ \delta_{i-1} = 0$, i.e. $\mathrm{Im}(\delta_i) \subseteq \mathrm{Ker}(\delta_{i-1})$. Complex is exact iff equality holds
- The determinant of the complex is

$$\det(K_\bullet) = \frac{\cdots \det(\delta_3)\det(\delta_1)\det(\delta_{-1})\cdots}{\cdots \det(\delta_4)\det(\delta_2)\det(\delta_0)\cdots}$$

for some maximal minors of δ_i's.

Theorem [Gelfand-Kapranov-Zelevinski]

$K_\bullet$ is exact $\iff$ $\mathcal{R}(f_0, \ldots, f_n) \neq 0$ $\iff$ $f_0 = \cdots = f_n = 0$ impossible

The terms in the Weyman complex

$$0 \to K_{n+1} \to \cdots \to K_2 \to K_1 \xrightarrow{\delta_1} K_0 \to \cdots \to K_{-n} \to 0$$

- The space of homogeneous polynomials of degree α_k
  $H^0(\mathbb{P}^{\ell_k}, \alpha_k) \neq 0 \Leftrightarrow \alpha_k \geq 0$
- Higher order cohomologies: $H^{\ell_k}(\mathbb{P}^{\ell_k}, \alpha_k) \neq 0 \Leftrightarrow \alpha_k < -\ell_k$
- Duality: $H^{\ell_k}(\alpha_k) \simeq H^0(-\ell_k - 1 - \alpha_k)^*$

$$K_\nu = \bigoplus_{p=0}^{n+1} K_{\nu,p} = \bigoplus_{p=0}^{n+1} \bigoplus_{i_1<\cdots<i_p} H^{p-\nu}\left(\boldsymbol{m} - \sum_{\theta=1}^{p} s_{i_\theta} \boldsymbol{d}\right)$$

Künneth formula:

$$H^{p-\nu}(\boldsymbol{u}) = \bigotimes_{k=1}^{r} H^{j_k}(\mathbb{P}^{\ell_k}, u_k)$$

$j_k \in \{0, \ell_k\}$, $j_1 + \cdots + j_r = p - \nu$

Formulae

Determinantal formula

If for some choice of $\boldsymbol{m} \in \mathbb{Z}^r$ the nonzero part is

$$0 \to K_i \xrightarrow{\delta_i} K_{i-1} \to 0$$

then $\boxed{\mathcal{R} = \det(M)}$. The matrix M of δ_i in some basis is square.

Objective: Find $\boldsymbol{m} \in \mathbb{Z}^r$ such that $K_\bullet$ has 2 non-zero terms

Bounds for all determinantal vectors

$$\max\{-d_k, -l_k\} \leq m_k \leq d_k \sum_0^n s_i - 1 + \min\{d_k - \ell_k, 0\}$$

bounds are tight w.r.t. individual coordinates[Dickenstein-Emiris03]

Example: The homogeneous case ($r = 1$)

Base degree $d = 1$, Scalar factors $\boldsymbol{s} = (s_0, s_1, \ldots, s_n)$.

$$\cdots \to K_2 \to K_1 \xrightarrow{\delta_1} K_0 \to K_{-1} \to \cdots$$

$$\cdots \to C_2 \to C_1 \oplus C_{n+1} \xrightarrow{\delta_1} C_0 \oplus C_n \to C_{n-1} \to \cdots$$

$$C_2 = \bigoplus_{i,j} H^0(m - s_i - s_j)$$

$$C_{n-1} = \bigoplus_{i_i,\ldots,i_{n-1}} H^n(m - s_{i_1} - \cdots - s_{i_{n-1}})$$

- Space of homogeneous polynomials of degree $m - t$:
  $\boxed{H^0(m - t) \neq 0 \Leftrightarrow m - t \geq 0}$
- The dual space: $\boxed{H^n(m - t) \neq 0 \Leftrightarrow m - t < -n}$
- $H^0(m - t) = H^n(m - t) = 0 \iff m < t \leq m + n$

Example: The homogeneous case ($r = 1$)

Case $r = d = n = 1$, Scalar factors $\boldsymbol{s} = (1, 2)$.

$$0 \to C_2 \to C_1 \oplus C_2 \xrightarrow{\delta_1} C_0 \oplus C_1 \to C_0 \to 0$$

$$C_2 = H^0(m - 3)$$
$$C_0 = H^1(m - 0)$$

- Space of homogeneous polynomials of degree $m - t$: $\boxed{H^0(m - t) \neq 0 \Leftrightarrow m - t \geq 0}$
- The dual space: $\boxed{H^n(m - t) \neq 0 \Leftrightarrow m - t < -n}$
- $H^0(m - t) = H^n(m - t) = 0 \iff m < t \leq m + n$
- $C_2 = C_0 = 0 \iff m = -1, 0, 1, 2$

Characterize scaled determinantal data

Hybrid formula: e.g. $0 \to C_a \oplus C_{a'} \to C_b \oplus C_{b'} \to 0$

Characterize determinantal data $\ell, \boldsymbol{d}, \boldsymbol{s}$

$\exists$ a determinantal formula:

- If $\boldsymbol{s} = \mathbf{1}$: $\iff \ell_k - \lceil \ell_k / d_k \rceil \leq 2$ [Weyman-Zelevinski94]
- If $r = 1$: $\iff s_2 + \cdots + s_n - n < s_0 + s_1$ [D'Andrea-Dicken01]

☞ If $\boldsymbol{s} \neq \mathbf{1}$: $\iff d_k \sum_{n-\pi[k]+2}^{n} s_i - l_k < d_k \sum_{0}^{\pi[k-1]+1} s_i, \ \forall k = 1, .., r$

Corollary

For any permutation $\pi : [1, r] \to [1, r]$, the vectors $m \in \mathbb{Z}^r$ contained in the box

$$d_k \sum_{n-\pi[k]+2}^{n} s_i - l_k \leq m_k \leq d_k \sum_{0}^{\pi[k-1]+1} s_i - 1$$

for $k = 1, \ldots, r$ are determinantal.

Pure formulae

Pure formula: $0 \to C_a \to C_b \to 0$

Characterize pure determinantal data $\ell, \boldsymbol{d}, \boldsymbol{s}$

$\exists$ pure determinantal formula:

- If $\boldsymbol{s} = \mathbf{1}$: $\iff \ell_k - \lceil \ell_k / d_k \rceil = 0$
  [Dickenstein-Emiris03, Sturmfels-Zelevinski94]

☞ If $\boldsymbol{s} \neq \mathbf{1}$: $\iff n = 1$ or $\boldsymbol{\ell} = (1, 1)$

Pure Bézout-type formulae are possible only if $\boldsymbol{s} = \mathbf{1}$.

Explicit choices (for pure formulae)

- $n = 1$: $m = d \sum_0^n s_i - 1$ or $m = -1$ (classic Sylvester)

☞ $\boldsymbol{\ell} = (1, 1)$: $\boldsymbol{m} = \left(-1, d_2 \sum_0^2 s_i - 1\right)$ or $\boldsymbol{m} = \left(d_1 \sum_0^2 s_i - 1, -1\right)$

Resultant matrices

- Find matrix M of $\delta_1 \;:\; K_1 \;\rightarrow\; K_0$ in standard monomial bases
- Block structure: $\delta_1 \;:\; \oplus_a C_a \;\rightarrow\; \oplus_b C_b$
- Construct a matrix for every restriction $\delta_{a,b} \;:\; C_a \;\rightarrow\; C_b$

Theorem [Weyman-Zelevinski94]

$\boxed{a - 1 < b \implies \delta_{a,b} = 0}$. Also, $\boxed{a - 1 = b \implies \delta_{a,b} \text{ is a Sylvester map}}$

Bézout-type constructions come from $a - 1 > b$.

Example (cont'd): Homogeneous case ($r = 1$)

The hybrid determinantal formula is: $0 \rightarrow C_1 \oplus C_{n+1} \xrightarrow{\delta_1} C_0 \oplus C_n \rightarrow 0$

$$M = \begin{array}{c} \\ C_1 \\ C_{n+1} \end{array} \begin{array}{c} \begin{array}{cc} C_0 & C_n \end{array} \\ \left[\begin{array}{cc} \boldsymbol{S} & \mathbf{0} \\ \boldsymbol{B} & \boldsymbol{S} \end{array}\right] \end{array}$$

Pure Sylvester-type matrices
scaled case

Ex.: The bilinear/biquadratic system $\ell = (1,1)$, $\boldsymbol{d} = (1,1)$, $\boldsymbol{s} = (1,1,2)$

$$\boldsymbol{m} = (3,-1) \implies C_2 \to C_1$$

$$\begin{aligned} f_0 &= a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 \\ f_1 &= b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2 \\ f_2 &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_1 x_2 + c_4 {x_1}^2 + c_5 {x_1}^2 x_2 + \\ &\quad + c_6 {x_2}^2 + c_7 x_1 {x_2}^2 + c_8 {x_1}^2 {x_2}^2 \end{aligned}$$

$$H^1(1,-3) \oplus H^1(0,-4) \oplus H^1(0,-4) \longrightarrow H^1(2,-2) \oplus H^1(2,-2) \oplus H^1(1,-3)$$

$$[\, g_0,\ g_1,\ g_2 \,] \begin{array}{c} {\scriptstyle f_0,f_1} \\ {\scriptstyle f_0,f_2} \\ {\scriptstyle f_1,f_2} \end{array} \overset{\begin{array}{ccc} f_0 & f_1 & f_2 \end{array}}{\begin{bmatrix} -M(f_1) & M(f_0) & \mathbf{0} \\ -M(f_2) & \mathbf{0} & M(f_0) \\ \mathbf{0} & -M(f_2) & M(f_1) \end{bmatrix}} = [-g_0 f_1 - g_1 f_2 \,,\ g_0 f_0 - g_2 f_2 \,,\ g_1 f_0 + g_2 f_1]$$

$$M(f_i) \ : \quad S\big(\boldsymbol{u} - \mathbf{deg} f_i\big) \ni \ g \ \longmapsto \ g f_i \ \in S(\boldsymbol{u})$$

Implementing multihomogeneous resultants in MAPLE

Extends implementation of unmixed case [Dickenstein-Emiris03]

- Computes all determinantal vectors, along with matrix dimension
- Computes the underlying complex of any $\boldsymbol{m}$-vector:

$$H^{p-\nu}(\boldsymbol{m} - z\boldsymbol{d}) \leftrightarrow \left(\{k_1, \ldots, k_t\}, \{i_1, \ldots, i_p\}\right)$$

- Constructs the (hybrid) matrix block by block.

```
> read mhomo-scaled.mpl:
> l:=vector([1,1]):  d:=l:  s:= vector([1,1,2]):
> f:= Makesystem(l,d,s):
> read multires.mpl:
> M:=mresultant( f, [x1, x2] ):  det(M);
                    0
> B:= mbezout( f, [x1, x2] ):  det(B);
                    0
> S:=makematrix(l,d,s,[3,-1]):evalb(det(S)=0);
                  false
```

References

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WZ94 J. Weyman and A. Zelevinsky. Multigraded formulae for multigraded resultants. *J. Algebr. Geom.,* 3(4):569-597, 1994

SZ94 B. Sturmfels and A. Zelevinsky. Multigraded resultants of Sylvester type. *J. of Algebra,* 163(1):115-127, 1994

Thank you !