Submersive Rational Difference Systems and their Accessibility

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Outline

1. Ring of shift operators
2. Motivation
3. Submersive systems and associated inversive fields
4. Accessibility for discrete-time systems
5. Remarks

Ring of shift operators

K : a field of characteristic zero; $\sigma : K \to K$ homomorphism.

1. (K, σ) is called a difference field;

   (K, σ) is inversive if $\sigma \in \mathrm{Aut}(K)$;

2. Define a ring $K[z;\sigma]$ with the commutation rule:

$$\forall r \in K, \ zr = \sigma(r)z;$$

3. Can define and compute greatest common left divisors (gcld) in $K[z; \sigma]$, provided $\sigma \in \mathrm{Aut}(K)$.

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Example

Let $K = \mathbb{Q}(t_0, t_1, t_2, \ldots)$ and $\sigma : t_i \mapsto t_{i+1}$ for all $i \in \mathbb{N}$.

- $t_1 z = z\, t_0 \implies \mathrm{gcld}(t_1 z,\, z) = \mathrm{gcld}(z\, t_0,\, z) = z$
- RightIdeal$(t_0 z,\, z)$ is not principal $\implies \mathrm{gcld}(t_0 z,\, z)$ does not exist.

Motivation

Set $u_i = u(t+i)$ and $y_j = y(t+j)$.

Input: u_0; output: y_0;

i/o eqn. $$y_n = f(u_0, u_1, \ldots, u_{n-1}, y_0, y_1, \ldots, y_{n-1})$$

$\Downarrow$ *differentiating*

1-form $$\mathrm{d}y_n = \sum_{i=0}^{n-1} \frac{\partial f}{\partial u_i} \mathrm{d}u_i + \sum_{j=0}^{n-1} \frac{\partial f}{\partial y_j} \mathrm{d}y_j.$$

$\Downarrow$ *rewriting by* $\begin{cases} \mathrm{d}u_i \to z^i \mathrm{d}u_0 \\ \mathrm{d}y_j \to z^j \mathrm{d}y_0 \end{cases}$

opr. $$P(z)\mathrm{d}y_0 = Q(z)\mathrm{d}u_0$$

where $P = z^n - \sum_{j=0}^{n-1} \frac{\partial f}{\partial y_j} z^j$, $Q = \sum_{i=0}^{n-1} \frac{\partial f}{\partial u_i} z^i$.

Accessibility and transfer functions

Control theory:

- The given i/o equation is accessible iff

$$\mathrm{gcld}(P, Q) = 1 \quad \text{mod the i/o eqn.}$$

- $Q^{-1}P$ is called the transfer function of the i/o equation.

Question: How does one compute $\mathrm{gcld}(P, Q)$ modulo an i/o equation?

For $f \in K(u_0, u_1, \ldots, y_0, y_1, \ldots, y_{n-1}) =: \bar{K}$

$$y_n = f$$

$$\Downarrow ?$$

$$\begin{array}{llll} \bar{\sigma}: & \bar{K} & \longrightarrow & \bar{K} \\ & u_i & \mapsto & u_{i+1} \quad \forall i \in \mathbb{N} \\ & y_j & \mapsto & y_{j+1} \quad \forall j \in \{0, 1, \ldots, n-2\} \\ & y_{n-1} & \mapsto & f. \end{array}$$

$$\Downarrow$$

Construct the inversive closure $\left(\widehat{K}, \hat{\sigma}\right)$ of $(\bar{K}, \bar{\sigma})$

$$\Downarrow$$

Compute $\operatorname{gcld}(P, Q)$ in $\widehat{K}[z; \hat{\sigma}]$.

Accessibility for state descriptions

System: u and x_j are functions in t, and $\sigma : t \mapsto t+1$

$$\left\{\begin{array}{ccc} \sigma(x_1) & = & f_1(u, x_1, \ldots, x_n) \\ \vdots & \vdots & \vdots \\ \sigma(x_n) & = & f_n(u, x_1, \ldots, x_n) \\ & & \\ y & = & g(x_1, \ldots, x_n) \end{array}\right.$$

How does one check its accessibility and compute its transfer function?

Given a difference system

$$\{\sigma(x_1) = f_1,\ \ldots,\ \sigma(x_n) = f_n\}$$

where $f_1, \ldots, f_n \in K(u(t), x_1(t), \ldots, x_n(t)) =: \bar{K}$

$$\Downarrow ?$$

$$\begin{array}{llll} \bar{\sigma}: & \bar{K} & \longrightarrow & \bar{K} \\ & u_i & \mapsto & u_{i+1} \quad \forall i \in \mathbb{N} \\ & x_j & \mapsto & f_j \quad \forall j \in \{1, \ldots, n\} \end{array}$$

$$\Downarrow$$

Construct the inversive closure $\left(\widehat{K}, \hat{\sigma}\right)$ of $(\bar{K}, \bar{\sigma})$

$$\Downarrow$$

Proposition (I)

The system is accessible iff the module of differential one-forms over $\widehat{K}[z; \hat{\sigma}]$ is torsion-free.

Submersive systems

(Grizzle 1993, Aranda-Bricaire, Kotta and Moog 1995)

Assume

- $\sigma \in \mathrm{Aut}(K)$;
- $\mathbf{U} = \{u_{i,\ell} \,|\, i = 1, \ldots, m, \ \ell \in \mathbb{N}\}$ and $\{x_1, \ldots, x_n\}$ indets over K
- $f_1, \ldots, f_n \in K(\mathbf{U}, x_1, \ldots, x_n)$.

A rational difference system

$$\{\sigma(x_1) = f_1(\mathbf{U}, x_1, \ldots, x_n), \ \ldots \ \sigma(x_n) = f_n(\mathbf{U}, x_1, \ldots, x_n)\}$$

is said to be submersive if

$$\mathrm{rank}\left(\frac{\partial(f_1, \ldots, f_n)}{\partial(u_{1,0} \ldots, u_{m,0}, \ x_1, \ldots, x_n)}\right) = n.$$

Proposition (II)

Let $\bar{K} = K(\mathbf{U}, x_1, \ldots, x_n)$. *The system*

$$\{\sigma(x_1) = f_1, \ldots, \sigma(x_n) = f_n\}$$

is submersive iff σ *can be extended to* $\bar{\sigma}$ *s.t.*

$$\begin{array}{ccc} K & \hookrightarrow & \bar{K} \\ \downarrow \sigma & \circlearrowright & \downarrow \bar{\sigma} \\ K & \hookrightarrow & \bar{K} \end{array} \quad \text{with} \quad \begin{array}{rcl} \bar{\sigma}\left(u_{i,\ell}\right) & = & u_{i,\ell+1} \\ \bar{\sigma}\left(x_j\right) & = & f_j. \end{array}$$

Example

Let $f \in K(\mathbf{U}, x_1, x_2)$ with $f \neq 0$. The system

$$\{\sigma(x_1) = f,\ \sigma(x_2) = -f\}$$

is not submersive.

Suppose

$$\begin{array}{rcl} \bar{\sigma}: \quad K(\mathbf{U}, x_1, x_2) & \rightarrow & K(\mathbf{U}, x_1, x_2) \\ u_i & \mapsto & u_{i+1} \\ x_1 & \mapsto & f \\ x_2 & \mapsto & -f \\ & \Downarrow & \\ \bar{\sigma}(x_1 + x_2) & = & 0 \\ & \Downarrow & \\ \bar{\sigma}\left(\frac{1}{x_1+x_2}\right) & \text{not} & \text{defined.} \end{array}$$

Proposition (III)

A rational difference system

$$\{\sigma(x_1) = f_1(\mathbf{U}, x_1, \ldots, x_n) \ldots, \sigma(x_n) = f_n(\mathbf{U}, x_1, \ldots, x_n)\} \qquad (*)$$

is submersive iff $(*)$ *corresponds to a proper, prime and reflexive difference ideal*

- $(*)$ has a generic solution
- $(\bar{K}, \bar{\sigma}) \cong K \langle \text{generic solution of } (*) \rangle$.

Call $(\bar{K}, \bar{\sigma})$ the difference field associated to $(*)$.

Submersive equations

Set $u_i = \sigma^i(u)$ and $y_j = \sigma^j(y)$.

Assume

- $\sigma \in \mathrm{Aut}(K)$;
- $\mathbf{U} = \{u_\ell \,|\, \ell \in \mathbb{N}\}$ and $\{y_0, y_1, \ldots, y_{n-1}\}$ indets over K
- $f \in K\,(\mathbf{U},\, y_0, y_1, \ldots, y_{n-1})$.

A difference equation

$$y_n = f$$

is said to be submersive if

either u_0 or y_0 appears in f.

Corollary

Let $\bar{K} = K(\mathbf{U}, y_0, y_1, \ldots, y_{n-1})$*.*
Then $y_n = f$ *is submersive iff* σ *can be extended to* $\bar{\sigma}$ *s.t.*

$$\begin{array}{ccc} K & \hookrightarrow & \bar{K} \\ \downarrow \sigma & \circlearrowright & \downarrow \bar{\sigma} \\ K & \hookrightarrow & \bar{K} \end{array} \quad \text{with} \quad \begin{array}{cccl} u_\ell & \mapsto & u_{\ell+1} & \ell \in \mathbb{N} \\ y_j & \mapsto & y_{j+1} & 0 \le j \le n-2 \\ y_{n-1} & \mapsto & f & \end{array}$$

Call $(\bar{K}, \bar{\sigma})$ the difference field associated to $y_n = f$.

Inversive closures (Cohn 1965)

Let $(\bar{K}, \bar{\sigma})$ be a difference field.

A difference field $(\widehat{K}, \hat{\sigma})$ is called the inversive closure of $(\bar{K}, \bar{\sigma})$ if

- $\hat{\sigma} \in \mathrm{Aut}(\widehat{K})$;
- $\widehat{K} \supseteq \bar{K}$ and $\hat{\sigma}|_{\bar{K}} = \bar{\sigma}$;
- $\forall a \in \widehat{K}, \exists \ell \in \mathbb{N}, \ \hat{\sigma}^{\ell}(a) \in \bar{K}$

Example

Consider a submersive equation

$$y_2 = \frac{y_0}{u_0 u_1} \quad \text{over } \mathbb{R}.$$

Let $\mathbf{U} = \{u_0, u_1, u_2, \ldots,\}$.

associated field: $\bar{K} = K(\mathbf{U}, y_0, y_1)$
$\bar{\sigma}(u_i) = u_{i+1},\ \bar{\sigma}(y_0) = y_1,\ \bar{\sigma}(y_1) = \frac{y_0}{u_0 u_1}$

inversive closure: $\widehat{K} = \bar{K}(u_{-1}, u_{-2}, \ldots)$
$\hat{\sigma}(u_i) = u_{i+1}$ with $i \leq 0$.

Note: $\hat{\sigma}(u_{-1} u_0 y_1) = y_0 \Longrightarrow \hat{\sigma}$ is surjective.

Tower:

$$(\mathbb{R}, \mathbf{1}) \subset (\bar{K}, \bar{\sigma}) \subset \left(\widehat{K}, \hat{\sigma}\right)$$

Check accessibility (i/o equations)

Given a submersive i/o equation:

$$y_n = f\left(u_0, u_1 \ldots, u_{n-1},\ y_0, y_1, \ldots, y_{n-1}\right),$$

where f is rational.

1. $y_n = f \xrightarrow{\mathrm{d}} P(z)\mathrm{d}y_0 = Q(z)\mathrm{d}u_0$
2. Construct the associated inversive field $\left(\widehat{K},\ \hat{\sigma}\right)$.
3. Compute $\mathrm{gcld}(P, Q)$ in $\widehat{K}[z;\ \widehat{\sigma}]$.

$$\mathrm{gcld}(P,\ Q) = 1 \text{ iff } y_n = f \text{ is accessible.}$$

Example

Put $u_i = x(t+i)$ and $y_j = y(t+j)$, and consider:

$$(u_1^3)\, y_3 = -u_1^2 y_1^2 - u_2 y_0^4 - 2u_0 u_2 y_2 y_0^2 - u_2 u_0^2 y_2^2. \quad (*)$$

1. $P\,\mathrm{d}y = Q\,\mathrm{d}u$, where

$$\begin{aligned} P &= z^3 + \frac{2u_2u_0(y_0^2+u_0y_2)}{u_1^3} z^2 + \frac{2y_1}{u_1} z + \frac{4u_2y_0(y_0^2+u_0y_2)}{u_1^3} \\ Q &= -\frac{(y_0^2+u_0y_2)^2}{u_1^3} z^2 + \frac{u_1^2y_1^2+3u_2y_0^4+3u_2u_0^2y_2^2+6u_0u_2y_2y_0^2}{u_1^4} z - \frac{2u_2y_2(y_0^2+u_0y_2)}{u_1^3}. \end{aligned}$$

2. Construct $(\widehat{K}, \widehat{\sigma})$.
3. In $\hat{K}[z;\hat{\sigma}]$: $\mathrm{gcld}(P,\ Q) = z + \frac{2u_0u_2(y_0^2+u_0y_2)}{u_1^3} \Longrightarrow$ not accessible.

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1. $P\,\mathrm{d}y = Q\,\mathrm{d}u$, where

$$\begin{aligned} P &= z^3 + \frac{2u_2u_0(y_0^2+u_0y_2)}{u_1^3} z^2 + \frac{2y_1}{u_1} z + \frac{4u_2y_0(y_0^2+u_0y_2)}{u_1^3} \\ Q &= -\frac{(y_0^2+u_0y_2)^2}{u_1^3} z^2 + \frac{u_1^2y_1^2+3u_2y_0^4+3u_2u_0^2y_2^2+6u_0u_2y_2y_0^2}{u_1^4} z - \frac{2u_2y_2(y_0^2+u_0y_2)}{u_1^3}. \end{aligned}$$

2. Construct $(\widehat{K}, \widehat{\sigma})$.
3. In $\hat{K}[z;\hat{\sigma}]$: $\mathrm{gcld}(P,\ Q) = z + \dfrac{2u_0u_2(y_0^2+u_0y_2)}{u_1^3} \Longrightarrow$ not accessible.

$$(*) \Longleftrightarrow \left\{ w_1 = -w^2, \quad w = \frac{y_2u_0 + y_0^2}{u_1} \right\}$$

Check accessibility (state description)

A submersive system

$$\{\sigma(x_1) = f_1, \ldots, \sigma(x_n) = f_n\}$$

where $f_1, \ldots, f_n$ are rational in $u, x_1, \ldots, x_n$.

Proposition (IV)

The system is accessible iff the $n \times n$ matrix

$$\left(\hat{\sigma}^{-1}(B),\ \hat{\sigma}^{-1}(A)\hat{\sigma}^{-2}(B),\ \ldots,\ \hat{\sigma}^{-1}(A)\cdots\hat{\sigma}^{-(n-1)}(A)\hat{\sigma}^{-n}(B) \right)$$

has rank n, where

$$A = \frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} \quad \text{and} \quad B = \left(\frac{\partial f_1}{\partial u}, \ldots, \frac{\partial f_n}{\partial u}\right)^T.$$

Remarks

1. Presented an algebraic setting for checking accessibility of rational discrete-time systems.

   (easier for continuous-time systems)

2. In the future

   - efficient gcld-computation in both differential and difference cases [Churchill and Zhang, to appear in J. Algebra]
   - multiple-input and multiple-output systems [Cheng and Labahn, JSC, 2006]
   - finite dynamic systems
   - irrational difference systems