* This work was supported by the Austrian Science Foundation (FWF) under the project DIFFOP (P20 336–N18).

A Skew Polynomial Approach to Integro-Differential Operators

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2009 International Symposium on Symbolic and Algebraic Computation (ISSAC)

Overview

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Goals

We want to deal **symbolically** with **boundary value problems**.

For the formulation of boundary value problems one needs:

- Differential equations like $u'' = f$.
- Evaluations express boundary conditions like $u(0) = u(1) = 0$.
- Integrals are needed to represent solutions (*Green's operators*) and Stieltjes boundary conditions.

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Integro-differential algebras

An ***ordinary integro-differential algebra*** is a K-algebra $\mathcal{F}$ together with a **derivation** $\partial\colon \mathcal{F} \to \mathcal{F}$ satisfying $\ker \partial = K$ and an **integral** $\int\colon \mathcal{F} \to \mathcal{F}$.

A derivation is a K-linear map with

$$\partial(fg) = \partial(f)g + f\partial(g) \quad \text{for all} \quad f, g \in \mathcal{F}$$

An integral is a K-linear map with

$$\partial \textstyle\int = 1$$

and, for all $f, g \in \mathcal{F}$,

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Standard example

An **standard example** for an ordinary integro-differential algebra is

- $K = \mathbb{R}$,
- $\mathcal{F} = \mathbb{R}[x]$,
- $\partial = d/dx$ and
- $\int = \int_c^x$

where $c \in \mathbb{R}$ is the **constant of integration.**

In the above example, the evaluation at c is given by

$$(1 - \textstyle\int\partial)(f) = f(x) - \int_c^x \frac{df}{dx}(\xi)\,\mathrm{d}\xi = f(c).$$

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Construction by Gröbner bases

Given an integro-differential algebra $(\mathcal{F}, \partial, \int)$ we can construct the corresponding **integro-differential operators** $\mathcal{F}[\partial, \int]$ as quotient of the free algebra and the two-sided ideal generated by

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\begin{aligned}
gf &= g \cdot f & \partial f &= f\partial + \partial(f) \\
\mathtt{E}^2 &= \mathtt{E} & \partial \mathtt{E} &= 0 \\
\mathtt{E} f &= \mathtt{E}(f) \cdot \mathtt{E} & \partial \textstyle\int &= 1
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\begin{aligned}
\textstyle\int f \int &= \textstyle\int(f) \cdot \int - \int \cdot \int(f) \\
\textstyle\int f \partial &= f - \textstyle\int \cdot \partial(f) - \mathtt{E}(f) \cdot \mathtt{E} \\
\textstyle\int f \mathtt{E} &= \textstyle\int(f) \cdot \mathtt{E}
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where $f, g \in \mathcal{F}$, and $\mathtt{E} = 1 - \int \partial$ is the evaluation.

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Skew polynomials

For a ring A with derivation $\delta\colon A \to A$, the skew polynomials $A[\xi;\delta]$ are defined as **polynomials in ξ** with the multiplication given by

$$\xi a = a\xi + \delta(a) \qquad \text{for all} \quad a \in A.$$

This models, e. g., differential operators where

$$\Big(\frac{d}{dx} \circ f\Big)(g) = \frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g = \Big(f \circ \frac{d}{dx} + \frac{df}{dx}\Big)(g)$$

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The integro-differential Weyl algebra

Analogously to the **Weyl algebra** we would like to model **integro-differential operators over the (usual) polynomials** $K[x]$ **in the variables** ∂ **and** $\ell = \int$ where K is a field.

We have to satisfy the relations

$$\partial x = x\partial + 1$$

and $\ell x = x\ell - \ell^2.$

But in the second relation the degree in ℓ is not preserved.

This problem can be overcome by changing the rôles of x and ∂, ℓ.

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$$\delta(\partial) = -1 \qquad \text{and} \qquad \delta(\ell) = \ell^2.$$

We have zero-divisors like, e. g., $\partial\mathrm{E} = 0$ where $\mathrm{E} = 1 - \ell\partial$ as before.

Every nonzero ideal of $K\langle\partial, \ell\rangle$ contains (E), and (E) is the only nontrivial δ-ideal.

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Taking $K\langle\partial,\ell\rangle$ as ground ring we define the ***integro-differential Weyl algebra*** to be

$$\mathrm{A}_1(\partial,\ell) = K\langle\partial,\ell\rangle[x;\delta].$$

This algebra satisfies

$$x\partial = \partial x - 1$$
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By restricting δ to $K[\ell]$ one obtains the **integro Weyl algebra**

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A K-basis is given by

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Isomorphisms

We have the **isomorphism of rings**

$$\frac{\mathrm{A}_1(\partial, \ell)}{(\mathrm{E})} \cong K[\partial, \partial^{-1}][x; \delta].$$

The integro-differential Weyl algebra becomes isomorphic to the integro-differential operators by fixing the constant of integration:

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We achieved

- **modelling boundary value problems** with **polynomial coefficients** using skew polynomials
- connecting this to previous approaches.

Further studies should be devoted to

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Thank you for your attention!