A Skew Polynomial Approach to Integro-Differential Operators

Johannes Middeke

(Joint work with Georg Regensburger and Markus Rosenkranz)

Research Institute for Symbolic Computation (RISC),
Johannes Kepler University,
Linz, Austria

2009 International Symposium on Symbolic and Algebraic Computation (ISSAC)

*This work was supported by the Austrian Science Foundation (FWF) under the project DIFFOP (P20 336–N18).*
Overview

1. Introduction
2. Previous work
3. The integro-differential Weyl algebra
4. Connecting the approaches
5. Conclusion
Overview

1 Introduction

2 Previous work

3 The integro-differential Weyl algebra

4 Connecting the approaches

5 Conclusion
Introduction

Goals

We want to deal \textit{symbolically} with \textbf{boundary value problems}.

For the formulation of boundary value problems one needs

- Differential equations like \( u'' = f \).
- Evaluations express boundary conditions like \( u(0) = u(1) = 0 \).
- Integrals are needed to represent solutions (Green's operator) — and Stieltjes boundary conditions.

The goal for today is a unified algebraic description of these three components.
Goals

We want to deal symbolically with boundary value problems.

For the formulation of boundary value problems one needs

- **Differential equations** like \( u'' = f \).
- Evaluations express boundary conditions like \( u(0) = u(1) = 0 \).
- Integrals are needed to represent solutions (Green's operator) — and Stieltjes boundary conditions.

The goal for today is a unified algebraic description of these three components.
Introduction

Goals

We want to deal symbolically with boundary value problems.

For the formulation of boundary value problems one needs

- Differential equations like $u'' = f$.
- **Evaluations** express boundary conditions like $u(0) = u(1) = 0$.
- Integrals are needed to represent solutions (Green's operator) — and Stieltjes boundary conditions.

The goal for today is a unified algebraic description of these three components.
Goals

We want to deal symbolically with boundary value problems.

For the formulation of boundary value problems one needs

- Differential equations like $u'' = f$.
- Evaluations express boundary conditions like $u(0) = u(1) = 0$.
- **Integrals** are needed to represent solutions (*Green’s operator*) — and Stieltjes boundary conditions.

The goal for today is a unified algebraic description of these three components.
Goals

We want to deal symbolically with boundary value problems.

For the formulation of boundary value problems one needs

- Differential equations like $u'' = f$.
- Evaluations express boundary conditions like $u(0) = u(1) = 0$.
- Integrals are needed to represent solutions (Green’s operator) — and Stieltjes boundary conditions.

The goal for today is a unified algebraic description of these three components.
Overview

1. Introduction
2. Previous work
3. The integro-differential Weyl algebra
4. Connecting the approaches
5. Conclusion
Integro-differential algebras

An ordinary integro-differential algebra is a $K$-algebra $\mathcal{F}$ together with a derivation $\partial : \mathcal{F} \rightarrow \mathcal{F}$ satisfying $\ker \partial = K$ and an integral $\int : \mathcal{F} \rightarrow \mathcal{F}$.

A derivation is a $K$-linear map with

\[ \partial(fg) = \partial(f)g + f \partial(g) \quad \text{for all } f, g \in \mathcal{F}. \]

An integral is a $K$-linear map with

\[ \partial \int = 1 \]

and, for all $f, g \in \mathcal{F}$,

\[ (\int f)(\int g) = \int(f \int g) + \int(g \int f). \]
Integro-differential algebras

An ordinary integro-differential algebra is a $K$-algebra $\mathcal{F}$ together with a derivation $\partial : \mathcal{F} \to \mathcal{F}$ satisfying $\ker \partial = K$ and an integral $\int : \mathcal{F} \to \mathcal{F}$.

A derivation is a $K$-linear map with

$$\partial(fg) = \partial(f)g + f\partial(g)$$

for all $f, g \in \mathcal{F}$.

An integral is a $K$-linear map with

$$\partial\int = 1$$

and, for all $f, g \in \mathcal{F}$,

$$(\int f)(\int g) = \int(f \int g) + \int(g \int f).$$
Integro-differential algebras

An ordinary integro-differential algebra is a $K$-algebra $\mathcal{F}$ together with a derivation $\partial : \mathcal{F} \rightarrow \mathcal{F}$ satisfying $\ker \partial = K$ and an integral $\int : \mathcal{F} \rightarrow \mathcal{F}$.

A derivation is a $K$-linear map with

$$\partial(fg) = \partial(f)g + f \partial(g) \quad \text{for all} \quad f, g \in \mathcal{F}.$$ 

An integral is a $K$-linear map with

$$\partial \int = 1$$

and, for all $f, g \in \mathcal{F}$,

$$(\int f)(\int g) = \int (f \int g) + \int (g \int f).$$
Standard example

An \textbf{standard example} for an ordinary integro-differential algebra is

- $K = \mathbb{R}$,
- $\mathcal{F} = \mathbb{R}[x]$,
- $\partial = d/dx$ and
- $\int = \int_c^x$

where $c \in \mathbb{R}$ is the \textbf{constant of integration}.

In the above example, the evaluation at $c$ is given by

$$(1 - \int \partial)(f) = f(x) - \int_c^x \frac{df}{d\xi}(\xi) \, d\xi = f(c).$$
Standard example

An standard example for an ordinary integro-differential algebra is

- $K = \mathbb{R}$,
- $\mathcal{F} = \mathbb{R}[x]$,  
- $\partial = d/dx$ and
- $\int = \int_c^x$

where $c \in \mathbb{R}$ is the constant of integration.

In the above example, the evaluation at $c$ is given by

$$(1 - \int \partial)(f) = f(x) - \int_c^x \frac{df}{dx} (\xi) \, d\xi = f(c).$$
Integro-differential operators

Construction by Gröbner bases

Given an integro-differential algebra \((\mathcal{F}, \partial, \int)\) we can construct the corresponding **integro-differential operators** \(\mathcal{F}[\partial, \int]\) as quotient of the free algebra and the two-sided ideal generated by

\[
\begin{align*}
gf &= g \cdot f \\
e^2 &= e \\
e f &= e(f) \cdot e \\
\int f \int &= \int(f) \cdot \int - \int \cdot \int(f) \\
\int f \partial &= f - \int \cdot \partial(f) - e(f) \cdot e \\
\int f e &= \int(f) \cdot e
\end{align*}
\]

where \(f, g \in \mathcal{F}\), and \(e = 1 - \int \partial\) is the evaluation.
Integro-differential operators
Construction by Gröbner bases

Given an integro-differential algebra \((\mathcal{F}, \partial, \int)\) we can construct the corresponding integro-differential operators \(\mathcal{F}[\partial, \int]\) as quotient of the free algebra and the two-sided ideal generated by

\[
\begin{align*}
gf &= g \cdot f \\
e^2 &= e \\
e f &= e(f) \cdot e \\
\int f \int &= \int(f) \cdot \int - \int \cdot \int(f) \\
\int f \partial &= f - \int \cdot \partial(f) - e(f) \cdot e \\
\int f e &= \int(f) \cdot e
\end{align*}
\]

where \(f, g \in \mathcal{F}\), and \(e = 1 - \int \partial\) is the evaluation.

The table describes an infinite two-sided non-commutative Gröbner basis.
Overview

1. Introduction
2. Previous work
3. The integro-differential Weyl algebra
4. Connecting the approaches
5. Conclusion
The integro-differential Weyl algebra

Skew polynomials

For a ring $A$ with derivation $\delta: A \to A$, the skew polynomials $A[\xi; \delta]$ are defined as polynomials in $\xi$ with the multiplication given by

$$\xi a = a\xi + \delta(a) \quad \text{for all} \quad a \in A.$$

This models, e.g., differential operators where

$$\left( \frac{d}{dx} \circ f \right)(g) = \frac{d}{dx}(fg) = \frac{dg}{dx} + \frac{df}{dx}g = \left( f \circ \frac{d}{dx} + \frac{df}{dx} \right)(g).$$

In general, the multiplication rule implies

$$\operatorname{deg}_\xi(fg) \leq \operatorname{deg}_\xi f + \operatorname{deg}_\xi g.$$
Skew polynomials

For a ring $A$ with derivation $\delta : A \to A$, the skew polynomials $A[\xi; \delta]$ are defined as polynomials in $\xi$ with the multiplication given by

$$\xi a = a\xi + \delta(a) \quad \text{for all} \quad a \in A.$$ 

This models, e.g., differential operators where

$$\left( \frac{d}{dx} \circ f \right)(g) = \frac{d}{dx} (fg) = f \frac{dg}{dx} + \frac{df}{dx} g = \left( f \circ \frac{d}{dx} + \frac{df}{dx} \right)(g).$$

In general, the multiplication rule implies

$$\deg_\xi (fg) \leq \deg_\xi f + \deg_\xi g.$$
Skew polynomials

For a ring $A$ with derivation $\delta: A \to A$, the skew polynomials $A[\xi; \delta]$ are defined as polynomials in $\xi$ with the multiplication given by

$$\xi a = a\xi + \delta(a) \quad \text{for all} \quad a \in A.$$ 

This models, e.g., differential operators where

$$\left( \frac{d}{dx} \circ f \right)(g) = \frac{d}{dx}(fg) = f \frac{dg}{dx} + \frac{df}{dx}g = \left( f \circ \frac{d}{dx} + \frac{df}{dx} \right)(g).$$

In general, the multiplication rule implies

$$\deg_\xi(fg) \leq \deg_\xi f + \deg_\xi g.$$
The integro-differential Weyl algebra

Analogously to the **Weyl algebra** we would like to model **integro-differential operators over the (usual) polynomials** $K[x]$ **in the variables** $\partial$ and $\ell = \int$ where $K$ is a field.

We have to satisfy the relations

$$\partial x = x\partial + 1$$

and

$$\ell x = x\ell - \ell^2.$$

But in the second relation the degree in $\ell$ is not preserved.

This problem can be overcome by changing the rôles of $x$ and $\partial, \ell$. 

J. Middeke (RISC)
The integro-differential Weyl algebra

Analogously to the Weyl algebra we would like to model integro-differential operators over the (usual) polynomials \( K[x] \) in the variables \( \partial \) and \( \ell = \int \) where \( K \) is a field.

We have to satisfy the relations

\[
\partial x = x\partial + 1 \quad \text{and} \quad \ell x = x\ell - \ell^2.
\]

But in the second relation the degree in \( \ell \) is not preserved.

This problem can be overcome by changing the rôles of \( x \) and \( \partial, \ell \).
Analogously to the Weyl algebra we would like to model integro-differential operators over the (usual) polynomials $K[x]$ in the variables $\partial$ and $\ell = \int$ where $K$ is a field.

We have to satisfy the relations

$$\partial x = x\partial + 1$$

and $$\ell x = x\ell - \ell^2.$$ 

But in the second relation the degree in $\ell$ is not preserved.

This problem can be overcome by changing the rôles of $x$ and $\partial, \ell$. 
The integro-differential Weyl algebra

Analogously to the Weyl algebra we would like to model integro-differential operators over the (usual) polynomials $K[x]$ in the variables $\partial$ and $\ell = \int$ where $K$ is a field.

We have to satisfy the relations

$$\partial x = x\partial + 1$$

and

$$\ell x = x\ell - \ell^2.$$ 

But in the second relation the degree in $\ell$ is not preserved.

This problem can be overcome by changing the rôles of $x$ and $\partial, \ell$. 
The ground ring

We take $K\langle \partial, \ell \rangle$ as ground ring where

$$\partial \ell = 1.$$ 

On $K\langle \partial, \ell \rangle$ we define a derivation $\delta$ by

$$\delta(\partial) = -1 \quad \text{and} \quad \delta(\ell) = \ell^2.$$ 

We have zero-divisors like, e.g., $\partial e = 0$ where $e = 1 - \ell \partial$ as before.

Every nonzero ideal of $K\langle \partial, \ell \rangle$ contains $(e)$, and $(e)$ is the only nontrivial $\delta$-ideal.
The ground ring

We take $K\langle \partial, \ell \rangle$ as ground ring where

$$\partial \ell = 1.$$ 

On $K\langle \partial, \ell \rangle$ we define a derivation $\delta$ by

$$\delta(\partial) = -1 \quad \text{and} \quad \delta(\ell) = \ell^2.$$ 

We have zero-divisors like, e.g., $\partial e = 0$ where $e = 1 - \ell \partial$ as before.

Every nonzero ideal of $K\langle \partial, \ell \rangle$ contains $(e)$, and $(e)$ is the only nontrivial $\delta$-ideal.
The ground ring

We take $K\langle \partial, \ell \rangle$ as ground ring where

$$\partial \ell = 1.$$ 

On $K\langle \partial, \ell \rangle$ we define a derivation $\delta$ by

$$\delta(\partial) = -1 \quad \text{and} \quad \delta(\ell) = \ell^2.$$ 

We have zero-divisors like, e.g., $\partial e = 0$ where $e = 1 - \ell \partial$ as before.

Every nonzero ideal of $K\langle \partial, \ell \rangle$ contains $(e)$, and $(e)$ is the only nontrivial $\delta$-ideal.
The ground ring

We take \( K\langle \partial, \ell \rangle \) as ground ring where
\[
\partial \ell = 1.
\]

On \( K\langle \partial, \ell \rangle \) we define a derivation \( \delta \) by
\[
\delta(\partial) = -1 \quad \text{and} \quad \delta(\ell) = \ell^2.
\]

We have zero-divisors like, e.g., \( \partial \varepsilon = 0 \) where \( \varepsilon = 1 - \ell \partial \) as before.

Every nonzero ideal of \( K\langle \partial, \ell \rangle \) contains \( (\varepsilon) \); and \( (\varepsilon) \) is the only nontrivial \( \delta \)-ideal.
The integro-differential Weyl algebra

Taking $K\langle \partial, \ell \rangle$ as ground ring we define the **integro-differential Weyl algebra** to be

$$A_1(\partial, \ell) = K\langle \partial, \ell \rangle[x; \delta].$$

This algebra satisfies

- $x \partial - 1$
- $x \ell = \ell x + \ell^2$
- and $\partial \ell = 1$.

One can show that $A_1(\partial, \ell)$ is neither simple nor (left or right) Noetherian.
The integro-differential Weyl algebra

Taking $K\langle \partial, \ell \rangle$ as ground ring we define the \textit{integro-differential Weyl algebra} to be

$$A_1(\partial, \ell) = K\langle \partial, \ell \rangle[x; \delta].$$

This algebra satisfies

$$x\partial = \partial x - 1$$

$$x\ell = \ell x + \ell^2$$

and

$$\partial \ell = 1.$$
The integro-differential Weyl algebra

Taking $K\langle \partial, \ell \rangle$ as ground ring we define the \emph{integro-differential Weyl algebra} to be

$$A_1(\partial, \ell) = K\langle \partial, \ell \rangle [x; \delta].$$

This algebra satisfies

$$x\partial = \partial x - 1,$$
$$x\ell = \ell x + \ell^2,$$
and $$\partial \ell = 1.$$

One can show that $A_1(\partial, \ell)$ is neither simple nor (left or right) Noetherian.
The integro Weyl algebra

By restricting $\delta$ to $K[\ell]$ one obtains the **integro Weyl algebra**

$$A_1(\ell) = K[\ell][x; \delta].$$

A $K$-basis is given by

$$\{ x^j \mid j \geq 0 \} \cup \{ x^m \ell x^n \mid m, n \geq 0 \}.$$

Also $A_1(\ell)$ is not simple.
The integro Weyl algebra

By restricting $\delta$ to $K[\ell]$ one obtains the integro Weyl algebra

$$A_1(\ell) = K[\ell][x; \delta].$$

A $K$-basis is given by

$$\{ x^j | j \geq 0 \} \cup \{ x^m \ell x^n | m, n \geq 0 \}.$$

Also $A_1(\ell)$ is not simple.
The integro Weyl algebra

By restricting $\delta$ to $K[\ell]$ one obtains the integro Weyl algebra

$$A_1(\ell) = K[\ell][x; \delta].$$

A $K$-basis is given by

$$\{ x^j \mid j \geq 0 \} \cup \{ x^m \ell x^n \mid m, n \geq 0 \}.$$

Also $A_1(\ell)$ is not simple.
Overview

1. Introduction
2. Previous work
3. The integro-differential Weyl algebra
4. Connecting the approaches
5. Conclusion
Isomorphisms

We have the **isomorphism of rings**

\[
\frac{A_1(\partial, \ell)}{(E)} \cong K[\partial, \partial^{-1}][x; \delta].
\]

The integro-differential Weyl algebra becomes isomorphic to the integro-differential operators by fixing the constant of integration:

\[
\frac{A_1(\partial, \ell)}{(Ex - cE)} \cong K[x][\partial, f].
\]
Isomorphisms

We have the isomorphism of rings

\[
\frac{A_1(\partial, \ell)}{(E)} \cong K[\partial, \partial^{-1}][x; \delta].
\]

The integro-differential Weyl algebra becomes isomorphic to the integro-differential operators by \textbf{fixing the constant of integration}:

\[
\frac{A_1(\partial, \ell)}{(E \times - cE)} \cong K[x][\partial, \int].
\]
Overview

1. Introduction
2. Previous work
3. The integro-differential Weyl algebra
4. Connecting the approaches
5. Conclusion
Final remarks

We achieved

- **modelling boundary value problems with polynomial coefficients** using skew polynomials
- connecting this to previous approaches.

Further studies should be devoted to

- adding more evaluations
- modelling partial differential operators

J. Middeke (RISC)  
ISSAC 2009
Final remarks

We achieved

- modelling boundary value problems with polynomial coefficients using skew polynomials
- connecting this to previous approaches.

Further studies should be devoted to

- adding more evaluations
- modelling partial differential operators
Final remarks

We achieved

- modelling boundary value problems with polynomial coefficients using skew polynomials
- connecting this to previous approaches.

Further studies should be devoted to

- adding more evaluations
- modelling partial differential operators
We achieved

- modelling boundary value problems with polynomial coefficients using skew polynomials
- connecting this to previous approaches.

Further studies should be devoted to

- adding more evaluations
- modelling \textit{partial differential operators}.\textit{
Thank you for your attention!