

A Skew Polynomial Approach to Integro-Differential Operators

Johannes Middeke

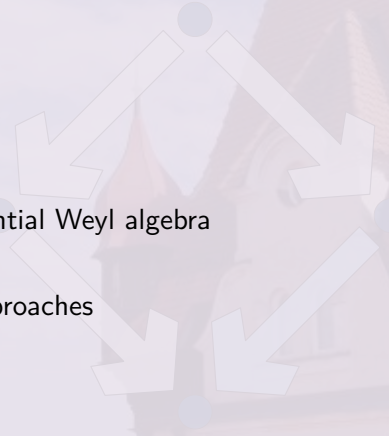
(Joint work with Georg Regensburger and Markus Rosenkranz)



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and Algebraic Computation (ISSAC)

Overview

- 1 Introduction
 - 2 Previous work
 - 3 The integro-differential Weyl algebra
 - 4 Connecting the approaches
 - 5 Conclusion
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Goals

We want to deal **symbolically** with **boundary value problems**.

For the formulation of boundary value problems one needs:

- ▶ Differential equations like $u'' = f$.
- ▶ Evaluations express boundary conditions like $u(0) = \alpha$.
- ▶ Integrals are needed to represent solutions (Green's operator) — and Stieltjes boundary conditions.

The goal for today is a unified algebraic description of these three components.

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- The diagram consists of five blue circular nodes arranged in a pentagonal pattern. Each node is connected to its adjacent neighbors by thick, white, blocky arrows. The arrow between the top and right nodes is highlighted in blue, corresponding to the second item in the list, 'Previous work'.

Integro-differential algebras

An **ordinary integro-differential algebra** is a K -algebra \mathcal{F} together with a **derivation** $\partial: \mathcal{F} \rightarrow \mathcal{F}$ satisfying $\ker \partial = K$ and an **integral** $\int: \mathcal{F} \rightarrow \mathcal{F}$.

A derivation is a K -linear map with

$$\partial(fg) = \partial(f)g + f\partial(g) \quad \text{for all } f, g \in \mathcal{F}.$$

An integral is a K -linear map with

and, for all $f, g \in \mathcal{F}$,

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$$\partial \int = 1$$

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Standard example

An **standard example** for an ordinary integro-differential algebra is

- ▶ $K = \mathbb{R}$,
- ▶ $\mathcal{F} = \mathbb{R}[x]$,
- ▶ $\partial = d/dx$ and
- ▶ $\int = \int_c^x$

where $c \in \mathbb{R}$ is the **constant of integration**.

In the above example, the evaluation map at c is given by

$$(1 - \int \partial)(f) \equiv f(x) - \int_c^x \frac{df}{dx}(\xi) d\xi = f(c)$$

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Integro-differential operators

Construction by Gröbner bases

Given an integro-differential algebra $(\mathcal{F}, \partial, \int)$ we can construct the corresponding **integro-differential operators** $\mathcal{F}[\partial, \int]$ as quotient of the free algebra and the two-sided ideal generated by

$gf = g \cdot f$	$\partial f = f\partial + \partial(f)$
$\mathbf{E}^2 = \mathbf{E}$	$\partial \mathbf{E} = 0$
$\mathbf{E}f = \mathbf{E}(f) \cdot \mathbf{E}$	$\partial \int = 1$
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$\int f \partial = f - \int \cdot \partial(f) - \mathbf{E}(f) \cdot \mathbf{E}$	
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where $f, g \in \mathcal{F}$, and $\mathbf{E} = 1 - \int \partial$ is the evaluation.

The table describes an infinite two-sided non-commutative Gröbner basis.

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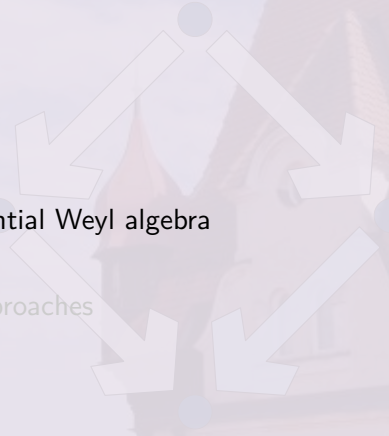
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Skew polynomials

For a ring A with derivation $\delta: A \rightarrow A$, the skew polynomials $A[\xi; \delta]$ are defined as **polynomials in ξ** with the multiplication given by

$$\xi a = a\xi + \delta(a) \quad \text{for all } a \in A.$$

This models, e. g., differential operators where

$$\left(\frac{d}{dx} \circ f\right)(g) = \frac{d}{dx}(fg) = \frac{dx}{dx}fg + \frac{df}{dx}g = \left(f \circ \frac{dx}{dx} + \frac{df}{dx}\right)g$$

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The integro-differential Weyl algebra

Analogously to the **Weyl algebra** we would like to model **integro-differential operators over the (usual) polynomials $K[x]$ in the variables ∂ and $\ell = \int$** where K is a field.

We have to satisfy the relations

$$[\partial, x] = \partial + 1$$

$$\text{and } [\partial, \ell] = \ell - \ell^2.$$

But in the second relation the degree in ℓ is not preserved.

This problem can be overcome by changing the rôles of x and ∂, ℓ .

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The ground ring

We take $K\langle \partial, \ell \rangle$ as **ground ring** where

$$\partial \ell = 1.$$

On $K\langle \partial, \ell \rangle$ we define a derivation δ by

$$\delta(\partial) = -1 \quad \text{and} \quad \delta(\ell) = \ell.$$

We have zero-divisors like, e. g., $\mathbb{E} \neq 0$ where $\mathbb{E} = 1 - \partial \ell$ as before.

Every nonzero ideal of $K\langle \partial, \ell \rangle$ contains (\mathbb{E}) , and (\mathbb{E}) is the only nonzero δ -ideal.

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The integro-differential Weyl algebra

Taking $K\langle\partial, \ell\rangle$ as ground ring we define the *integro-differential Weyl algebra* to be

$$A_1(\partial, \ell) = K\langle\partial, \ell\rangle[x; \delta].$$

This algebra satisfies

$$\begin{aligned} x\partial &= \partial x - 1 \\ \ell\partial &= \partial\ell + \ell^2 \\ \text{and } \partial\ell &= 1. \end{aligned}$$

One can show that $A_1(\partial, \ell)$ is neither simple nor (left or right) Noetherian.

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By restricting δ to $K[\ell]$ one obtains the **integro Weyl algebra**

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A K -basis is given by

$$\{x^j \mid j \geq 0\} \cup \{\ell^m x^n \mid m, n \geq 0\}$$

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Isomorphisms

We have the **isomorphism of rings**

$$\frac{A_1(\partial, \ell)}{(\mathbf{E})} \cong K[\partial, \partial^{-1}][x; \delta].$$

The integro-differential Weyl algebra becomes isomorphic to the ring of integro-differential operators by adding the constant of integration.

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$$\frac{A_1(\partial, \ell)}{(E)} \cong K[\partial, \partial^{-1}][x; \delta].$$

The integro-differential Weyl algebra becomes isomorphic to the integro-differential operators by **fixing the constant of integration**:

$$\frac{A_1(\partial, \ell)}{(E\mathbf{x} - cE)} \cong K[x][\partial, \int].$$

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Final remarks

We achieved

- ▶ **modelling boundary value problems with polynomial coefficients** using skew polynomials
- ▶ connecting this to previous approaches.

Further studies should be devoted to

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