Principal Intersection and Bernstein-Sato Polynomial of an Affine Variety

Viktor Levandovskyy, Daniel Andres — RWTH Aachen, Germany

Jorge Martín-Morales — University of Zaragoza, Spain

ISSAC, Seoul, 29 July 2009

Introduction and notations

Basic notations

- $\mathbb{K}$ stands for a field of characteristic 0 ($\mathbb{K} = \mathbb{C}$).
- $\mathbb{K}[s]$ the ring of polynomials in one variable over $\mathbb{C}$.
- $\mathbb{K}[x_1, \ldots, x_n]$ the ring of polynomials in n variables.
- $D_n := \mathbb{K}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$ the ring of $\mathbb{K}$-linear differential operators on $\mathbb{K}[x_1, \ldots, x_n]$, the n-th Weyl algebra:

$$\partial_i x_i = x_i \partial_i + 1, \quad \text{while for } i \neq j \ \partial_i x_j = x_j \partial_i, \ \partial_j \partial_i = \partial_i \partial_j, \ x_j x_i = x_i x_j$$

- $D_n[s] := D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$.

The $D_n[s]$-module $\mathbb{K}\left[x, s, \frac{1}{f}\right] \cdot f^s$

- Let $f \in \mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial.
- By $\mathbb{K}[x, s, \frac{1}{f}]$ we denote the ring of rational functions of the form

$$\frac{g(\mathbf{x}, s)}{f^r}$$

  where $g(\mathbf{x}, s) \in \mathbb{K}[x, s] = \mathbb{K}[x_1, \ldots, x_n, s]$.

- We denote by $\mathbb{K}[x, s, \frac{1}{f}] \cdot f^s$ the free $\mathbb{K}[x, s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s.
- $\mathbb{K}[x, s, \frac{1}{f}] \cdot f^s$ has a natural structure of left $D_n[s]$-module.

$$\partial_i \bullet f^s = s \frac{\partial f}{\partial x_i} \frac{1}{f} \cdot f^s \quad \in \quad \mathbb{K}\left[x, s, \frac{1}{f}\right] \cdot f^s$$

The global b-function a.k.a. Bernstein-Sato polynomial

Theorem (J. Bernstein, 1972)

*For a non-constant polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ there exists a linear partial differential operator $P(s)$ with polynomial coefficients in $\mathbb{C}[x_1, \ldots, x_n, s]$ **and** a univariate polynomial $b(s) \in \mathbb{C}[s]$, such that*

$$P(s) \bullet f^{s+1} = b(s) \cdot f^s,$$

Definition (Bernstein & Sato)

The set of all possible polynomials $b(s)$ satisfying the above equation is an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the Bernstein-Sato polynomial of f.

$$\mathrm{Ann}_{D[s]}(f^s) = \{Q(s) \in D[s] \mid Q(s) \bullet f^s = 0\} \subset D[s]$$

Reformulation of Bernstein's Theorem

Given $f \in \mathbb{K}[x]$ a polynomial, there exists an operator $P(s) \in D[s]$ and $b_f(s) \in \mathbb{K}[s]$, such that

$$P(s) \bullet f^{s+1} = b_f(s) \cdot f^s \iff P(s)f - b_f(s) \in \operatorname{Ann}_{D[s]} f^s.$$

The monic polynomial $b_f(s)$ of smallest degree, satisfying the above condition is the Bernstein-Sato polynomial.

Theorem (Kashiwara)

All roots of $b_f(s)$ are negative rational numbers.

Algorithm for computing Bernstein-Sato polynomial via $\operatorname{Ann}_{D[s]} f^s$

$$(\operatorname{Ann}_{D[s]} f^s + \langle f \rangle) \cap \mathbb{K}[s] = \langle b(s) \rangle$$

Evolution from OT to BM

1. **Oaku and Takayama** proved in 1999, that $\mathrm{Ann}_{D[s]}(f^s) =$

$$D\langle t, \partial_t \mid [\partial_t, t] = 1\rangle[u, v](\langle t - uf, uv - 1, \{u\frac{\partial f}{\partial x_i}\partial_t + \partial_i\}\rangle) \cap D[-t\partial_t - 1],$$

   "eliminate u, v and intersect the result with $D[s], s = -t\partial_t - 1$".

2. **Briançon and Maisonobe** proved in 2002, that $\mathrm{Ann}_{D[s]}(f^s) =$

$$D\langle \partial_t, s \mid [s, \partial_t] = \partial_t\rangle\langle f\partial_t + s, \{\frac{\partial f}{\partial x_i}\partial_t + \partial_i\}\rangle \cap D[s]$$

   "eliminate ∂_t".

3. F. Castro and J.-M. Ucha, JSC 2004 proved, that
   The BM method is less complex than the method of OT.

4. ISSAC 09: we give a new proof for the BM method, showing that
   The BM is a consequence of the OT method.

New Proof for Briançon and Maisonobe's Method

Ingredients

- G-algebras of Lie type (i. e. with relations $\forall 1 \leq i < j \leq n$ $x_j x_i = x_i x_j + d_{ij}$, $d_{ij} \in \mathbb{K}[x]$)
- generalized product criterion in G-algebras of Lie type (V. Levandovskyy and H. Schönemann, ISSAC 2003)
- preimage of a left ideal under a morphism of $\mathbb{K}$-algebras (V. Levandovskyy, ISSAC 2006)
- slim Gröbner basis algorithm (M. Brickenstein, 2005)

Adapted Preimage Theorem

Assume that $A = \mathbb{K}\langle \mathbf{x} \rangle / I_A$ and $B = \mathbb{K}\langle \mathbf{y} \rangle / I_B$ be G-algebras of Lie type and $\phi : A \to B$ be a homomorphism of $\mathbb{K}$-algebras. Then $A \otimes_{\mathbb{K}}^{\phi} B := \mathbb{K}\langle \mathbf{x}, \mathbf{y} \rangle / (I_A + I_B + \langle \{ y_j x_i - x_i y_j - y_j \phi(x_i) + \phi(x_i) y_j \} \rangle)$.
Suppose, that there exists an elimination ordering for B on $A \otimes_{\mathbb{K}} B$, satisfying $\mathrm{lm}(y_j \phi(x_i) - \phi(x_i) y_j) \prec x_i y_j$ for all $1 \le i \le n, 1 \le j \le m$.
Let A_1, B_1, C be G-algebras of Lie type and $\varphi : A_1 \to B_1$ be a homomorphism of $\mathbb{K}$-algebras. Consider the following data:

$$A = C \otimes_{\mathbb{K}} A_1, \quad B = C \otimes_{\mathbb{K}} B_1, \quad \phi = 1_C \otimes \varphi : A \to B,$$
$$E = A \otimes_{\mathbb{K}}^{\phi} B, \qquad E' = C \otimes_{\mathbb{K}} (A_1 \otimes_{\mathbb{K}}^{\varphi} B_1).$$

Then $E', A \otimes_{\mathbb{K}}^{\phi} B, E$ are G-alg of Lie type, and for a left ideal $J \subset B$

1. $(EI_\varphi + EJ) \cap E' = E' I_\varphi + E' J$.
2. $\phi^{-1}(J) = (E' I_\varphi + E' J) \cap A \subset (E' \cap A)$.

The latter intersection can be computed using Gröbner bases.

Theorem (Briançon and Maisonobe)

Let $S_p = \mathbb{K}\langle\{\partial t_j, s_j\} \mid \partial t_j s_k = s_k \partial t_j - \delta_{jk}\partial t_j\rangle$ *(the p-th shift algebra) and* $B = D_n \otimes_{\mathbb{K}} S_p$. *Moreover, consider the left ideal* $I \subset B$, *generated by* $\{s_j + f_j \partial t_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k\}$. *Then* $\mathrm{Ann}_{D_n[s]}(f^s) = I \cap D_n[s]$.

Setup

Let $A := D_n[s] = \mathbb{K}\langle\{s_j, x_i, \partial_i\} \mid \partial_i x_i = x_i \partial_i + 1\rangle$, and
$B := \mathbb{K}\langle\{t_j, \partial t_j, x_i, \partial_i\} \mid \{\partial_i x_i = x_i \partial_i + 1, \partial t_j t_j = t_j \partial t_j + 1\}\rangle$.
Thus in the notations of Preimage Theorem
$C = D_n, A_1 = \mathbb{K}[s], B_1 = \mathbb{K}\langle\{t_j, \partial t_j \mid \partial t_j t_j = t_j \partial t_j + 1\}\rangle$ and
$A = C \otimes A_1, B = C \otimes B_1$.

Consider $\varphi : A_1 \to B_1, \;\; s_j \mapsto -t_j \partial t_j - 1$, then
$I_\varphi = \langle\{t_j \partial t_j + s_j + 1\}\rangle \subset A_1 \otimes_{\mathbb{K}}^{\varphi} B_1 =: E'$.

Theorem (Briançon and Maisonobe)

Let $S_p = \mathbb{K}\langle\{\partial t_j, s_j\} \mid \partial t_j s_k = s_k \partial t_j - \delta_{jk}\partial t_j\rangle$ (the p-th shift algebra) and $B = D_n \otimes_{\mathbb{K}} S_p$. Moreover, consider the left ideal $I \subset B$, generated by $\{s_j + f_j \partial t_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i}\partial t_k\}$. Then $\mathrm{Ann}_{D_n[s]}(f^s) = I \cap D_n[s]$.

Setup

Let $A := D_n[s] = \mathbb{K}\langle\{s_j, x_i, \partial_i\} \mid \partial_i x_i = x_i\partial_i + 1\rangle$, and
$B := \mathbb{K}\langle\{t_j, \partial t_j, x_i, \partial_i\} \mid \{\partial_i x_i = x_i \partial_i + 1, \partial t_j t_j = t_j \partial t_j + 1\}\rangle$.
Thus in the notations of Preimage Theorem
$C = D_n, A_1 = \mathbb{K}[s], B_1 = \mathbb{K}\langle\{t_j, \partial t_j \mid \partial t_j t_j = t_j \partial t_j + 1\}\rangle$ and
$A = C \otimes A_1, B = C \otimes B_1$.

Consider $\varphi : A_1 \to B_1,\ \ s_j \mapsto -t_j\partial t_j - 1$, then
$I_\varphi = \langle\{t_j\partial t_j + s_j + 1\}\rangle \subset A_1 \otimes_{\mathbb{K}}^{\varphi} B_1 =: E'$.

Since $[t_k, s_j] = \delta_{jk} t_j$ and $[\partial t_k, s_j] = -\delta_{jk} \partial t_j$, the ordering conditions of Preimage Theorem are $t_j \prec s_j t_j, \partial t_j \prec s_j \partial t_j$. They are satisfied iff $1 \leq t_j, \partial t_j, s_j$.

By Preimage Theorem, for any $L \subset B$, $\phi^{-1}(L) = (I_\phi + L) \cap A$. We apply it to the Malgrange ideal

$$L = \langle \{ t_j - f_j, \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j \} \rangle.$$

Since $t_j \partial t_j + s_j + 1$ reduces to $f_j \partial t_j + s_j \in I_\phi + L$, we have

$$I_\phi + L = \langle \{ t_j - f_j, \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j, f_j \partial t_j + s_j \} \rangle.$$

Since $[t_k, s_j] = \delta_{jk} t_j$ and $[\partial t_k, s_j] = -\delta_{jk} \partial t_j$, the ordering conditions of Preimage Theorem are $t_j \prec s_j t_j, \partial t_j \prec s_j \partial t_j$. They are satisfied iff $1 \leq t_j, \partial t_j, s_j$.

By Preimage Theorem, for any $L \subset B$, $\phi^{-1}(L) = (I_\phi + L) \cap A$. We apply it to the Malgrange ideal

$$L = \langle \{ t_j - f_j, \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j \} \rangle.$$

Since $t_j \partial t_j + s_j + 1$ reduces to $f_j \partial t_j + s_j \in I_\phi + L$, we have

$$I_\phi + L = \langle \{ t_j - f_j, \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j, f_j \partial t_j + s_j \} \rangle.$$

Lemma

Consider an ordering $\prec_T$*, which satisfies the property* $\{t_j\} \gg \{x_i\}$*,* $\{\partial_i, s_j\} \gg \{x_i, \partial t_j\}$*. Moreover, set*

$$g_i := \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k, \ S_1 := \{t_j - f_j, g_i\},$$
$$S_2 := S_1 \cup \{s_j + f_j \partial t_j\} \subset E'.$$

Then S_1 *and* S_2 *are left Gröbner bases with respect to* $\prec_T$*.*

Proof: numerous applications of generalized product criterion.

Final Step

Since an ordering, eliminating $\{t_j\}$, satisfies the property in the Lemma, it remains to eliminate $\{\partial t_j\}$ from $S_2 \setminus \{t_i - f_i\}$, what is exactly the statement of the Theorem of Briançon and Maisonobe.

Computer algebra systems for D-modules

Computer algebra system SINGULAR has joined the club of systems, able to perform computations for *D*-module theory. To the best of our knowledge, this club consists of

- experimental program kan/sm1 by N. Takayama
- the package Dmodules in Macaulay2 by A. Leykin and H. Tsai
- bfct package in Risa/Asir by M. Noro
- there are rumours about ongoing development in CoCoA

The perfect package for *D*-modules must combine flexibility and rich functionality with high performance, e. g. in order to treat complicated examples, coming both from theory and applications.

Initial ideal with respect to weights

Definition

Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$. For a non-zero element of D, there exists $m \in \mathbb{R}$ such that

$$p = \sum_{\alpha,\beta \in \mathbb{N}_0^n:\ -w\alpha + w\beta = m} c_{\alpha\beta} x^\alpha \partial^\beta + \sum_{\alpha,\beta \in \mathbb{N}_0^n:\ -w\alpha + w\beta < m} c_{\alpha\beta} x^\alpha \partial^\beta$$

We call the graded ideal $\mathrm{in}_{(-w,w)}(I) := \mathbb{K} \cdot \{\mathrm{in}_{(-w,w)}(p) \mid p \in I\}$ the *initial ideal of I with respect to the weight w*.

Definition

Let $0 \neq w \in \mathbb{R}^n_{\geq 0}$ and $s := \sum_{i=1}^n w_i x_i \partial_i$. Then $\mathrm{in}_{(-w,w)}(I) \cap \mathbb{K}[s]$ is a principal ideal in $\mathbb{K}[s]$. Its monic generator $b(s)$ is called the *global b-function of I with respect to the weight w*.

Principal Intersection: Problem Formulation

A	associative $\mathbb{K}$-algebra
$J \subset A$	left ideal
G	finite Gröbner basis of J wrt. arbitrary global ordering
$s \in A \setminus \mathbb{K}$	any nonsonstant element

What can we say a priori about $J \cap \mathbb{K}[s]$?

There are the following situations:

1. $\operatorname{lm}(g) \nmid \operatorname{lm}(s^k)$ for all $g \in G, k \in \mathbb{N}_0$
2. Situation 1 does not hold, that is $\exists\, g \in G, k \geq 1$ with $\operatorname{lm}(g) \mid \operatorname{lm}(s^k)$
    1. $J \cdot s \subset J$ and $\dim_{\mathbb{K}}(\operatorname{End}_A(A/J)) < \infty$
    2. Situation 2.1 does not hold:
        1. ???

Principal Intersection: Problem Formulation

A	associative $\mathbb{K}$-algebra
$J \subset A$	left ideal
G	finite Gröbner basis of J wrt. arbitrary global ordering
$s \in A \setminus \mathbb{K}$	any nonsonstant element

What can we say a priori about $J \cap \mathbb{K}[s]$?

There are the following situations:

1. $\text{Im}(g) \nmid \text{Im}(s^k)$ for all $g \in G, k \in \mathbb{N}_0$
2. Situation 1 does not hold, that is $\exists\, g \in G, k \geq 1$ with $\text{Im}(g) \mid \text{Im}(s^k)$
    1. $J \cdot s \subset J$ and $\dim_{\mathbb{K}}(\text{End}_A(A/J)) < \infty$
    2. Situation 2.1 does not hold:
        1. ???

Principal Intersection: Problem Formulation

A	associative $\mathbb{K}$-algebra
$J \subset A$	left ideal
G	finite Gröbner basis of J wrt. arbitrary global ordering
$s \in A \setminus \mathbb{K}$	any nonsonstant element

What can we say a priori about $J \cap \mathbb{K}[s]$?

There are the following situations:

1. $\mathrm{lm}(g) \nmid \mathrm{lm}(s^k)$ for all $g \in G, k \in \mathbb{N}_0$
2. Situation 1 does not hold, that is $\exists\, g \in G, k \geq 1$ with $\mathrm{lm}(g) \mid \mathrm{lm}(s^k)$
   1. $J \cdot s \subset J$ and $\dim_{\mathbb{K}}(\mathrm{End}_A(A/J)) < \infty$
   2. Situation 2.1 does not hold:
      1. ???

Situations 1 und 2

Lemma (Situation 1)

If there exists no $g \in G$ such that $\operatorname{lm}(g)$ divides $\operatorname{lm}(s^k)$ for some $k \in \mathbb{N}_0$, then $J \cap \mathbb{K}[s] = 0$.

Situation 2

The converse to the lemma does not hold. Consider $J = \langle y^2 + x \rangle \subset \mathbb{K}[x, y]$. Then $J \cap \mathbb{K}[y] = \{0\}$ while $\{y^2 + x\}$ is a Gröbner basis of J for any ordering.

Situations 1 und 2

Lemma (Situation 1)

If there exists no $g \in G$ *such that* $\operatorname{lm}(g)$ *divides* $\operatorname{lm}(s^k)$ *for some* $k \in \mathbb{N}_0$*, then* $J \cap \mathbb{K}[s] = 0$*.*

Situation 2

The converse to the lemma does not hold. Consider $J = \langle y^2 + x \rangle \subset \mathbb{K}[x, y]$. Then $J \cap \mathbb{K}[y] = \{0\}$ while $\{y^2 + x\}$ is a Gröbner basis of J for any ordering.

Situation 2.1

Lemma

Let $J \cdot s \subset J$ and $\mathrm{End}_A(A/J)$ finite dimensional as $\mathbb{K}$-vector space. Then $J \cap \mathbb{K}[s] \neq 0$.

Proof:

- $\cdot s : A/J \to A/J,\ [a] \mapsto [a \cdot s]$ ist well-defined A-module endomorphism
- $\cdot s$ has minimal polynomial μ
- $\mu \in \mathbb{K}[s] \cap J$ and $\mu \neq 0$ (even $\mu \notin \mathbb{K}$) □

The condition $\dim_{\mathbb{K}} \mathrm{End}_A(A/J) < \infty$ is fulfilled, if

- A/J is finite dimensional over $\mathbb{K}$ or
- A is Weyl algebra and A/J is a holonomic A-module.

Situation 2.1

Lemma

Let $J \cdot s \subset J$ and $\mathrm{End}_A(A/J)$ finite dimensional as $\mathbb{K}$-vector space. Then $J \cap \mathbb{K}[s] \neq 0$.

Proof:

- $\cdot s : A/J \to A/J,\ [a] \mapsto [a \cdot s]$ ist well-defined A-module endomorphism
- $\cdot s$ has minimal polynomial μ
- $\mu \in \mathbb{K}[s] \cap J$ and $\mu \neq 0$ (even $\mu \notin \mathbb{K}$) □

The condition $\dim_{\mathbb{K}} \mathrm{End}_A(A/J) < \infty$ is fulfilled, if

- A/J is finite dimensional over $\mathbb{K}$ or
- A is Weyl algebra and A/J is a holonomic A-module.

Algorithm (pIntersect)

Input: $s \in A \setminus \mathbb{K}$, $J \subset A$ such that $J \cap \mathbb{K}[s] \neq \{0\}$
Output: $b \in \mathbb{K}[s]$ normed such, that $J \cap \mathbb{K}[s] = \langle b \rangle$

$G :=$ finite Gröbner basis of J wrt. *arbitrary* global ordering
$i := 1$
loop
if there exist $a_0, \ldots, a_{i-1} \in \mathbb{K}$ with

$$\mathrm{NF}(s^i, G) + \sum_{j=0}^{i-1} a_j \,\mathrm{NF}(s^j, G) = 0 \textbf{ then}$$

return $b := s^i + \sum_{j=0}^{i-1} a_j s^j$
else
$i := i + 1$
end if
end loop

Consequence and Improvement

Theorem

For a holonomic module D/I, a b-function of I wrt $(-w, w)$ is not constant.

Lemma

Let A be a $\mathbb{K}$-algebra, $J \subset A$ a left ideal and let $s \in A$. For $i \in \mathbb{N}$ put $r_i = \mathrm{NF}(s^i, J)$ and $q_i = s^i - r_i \in J$. Then we have for all $i \in \mathbb{N}$

$$r_{i+1} = \mathrm{NF}([s^i - r_i, r_1] + r_i r_1, J).$$

Note, that computing Lie bracket $[f, g]$ both in theory and in practice is easier and faster, than to compute $[f, g]$ as $f \cdot g - g \cdot f$.

Consequence and Improvement

Theorem

For a holonomic module D/I, a b-function of I wrt $(-w, w)$ is not constant.

Lemma

Let A be a $\mathbb{K}$-algebra, $J \subset A$ a left ideal and let $s \in A$. For $i \in \mathbb{N}$ put $r_i = \mathrm{NF}(s^i, J)$ and $q_i = s^i - r_i \in J$. Then we have for all $i \in \mathbb{N}$

$$r_{i+1} = \mathrm{NF}([s^i - r_i, r_1] + r_i r_1, J).$$

Note, that computing Lie bracket $[f, g]$ both in theory and in practice is easier and faster, than to compute $[f, g]$ as $f \cdot g - g \cdot f$.

Algorithm (`intersectUpTo`)

Input: $s_1, \ldots, s_r \in A \setminus \mathbb{K}$, $s_j s_i = s_i s_j$, $J \subset A$ left ideal, $k \in \mathbb{N}$

Output: Gröbner basis B of $J \cap \mathbb{K}[s_1, \ldots, s_r]$ up to degree k

$G :=$ Gröbner basis of J wrt. arbitrary global ordering

$d := 0, \qquad B := \emptyset$

while $d \leq k$ **do**

$M_d := \{s^\alpha \mid |\alpha| \leq d\}$

if $\exists\ a_m \in \mathbb{K}$ with $\sum_{m \in M_d} a_m \operatorname{NF}(m, G) = 0$ **then**

$f := \sum_{m \in M_d} a_m m$

if $f \neq 0$ and $f \notin \langle B \rangle$ **then**

$B := B \cup \{f\}$

end if

end if

$d := d + 1$

end while

return B

Fine tuning and Termination

- Found $p \in B$ with $\operatorname{lm}(p) = m$, ignore all multiples of m
- If G is a Gröbner basis of J wrt. $\prec$, ignore

$$\{m \in M_d \mid \max_{\prec}(m' \in L(G) \cap M_d) \prec m\},$$

since $p \in J \cap \mathbb{K}[s_1, \ldots, s_r] \Leftrightarrow \operatorname{lm}(p) \in L(G) \cap \mathbb{K}[s_1, \ldots, s_r]$.

- $\operatorname{NF}(m, G) = m$, if $m \notin L(G) \cap \mathbb{K}[s_1, \ldots, s_r]$.

- In general, the termination is provided in case when
  - $J \cap \mathbb{K}[s_1, \ldots, s_r]$ is zero-dimensional or
  - it is known in advance, that $J \cap \mathbb{K}[s_1, \ldots, s_r]$ is principal.

Computing Bernstein Data

- There are several methods to compute $\mathrm{Ann}_{\mathbf{D[s]}}\, \mathbf{f^s} \subset D[s]$ (BM etc.)
- For computing $\mathbf{b(s)} \in \mathbb{K}[s]$, one uses e.g.

  - $_{\mathbb{K}[s]}\langle b(s)\rangle = (\mathrm{Ann}_{D[s]}\, f^s + \langle f\rangle) \cap \mathbb{K}[s]$. Note, that this computation via elimination ranges from "very hard" to "hopeless" in practice.

  - $_{\mathbb{K}[s]}\langle b(s)\rangle = \mathrm{in}_{(-w,w)}(\langle t - f, \{\frac{\partial f}{\partial x_i}\partial_t + \partial_i\}\rangle) \cap \mathbb{K}[\theta]$ for $w \in \mathbb{R}_+^{n+1}$ and $\theta = w_0 t\partial_t + \sum_{i=1}^n w_i x_i \partial_i$.

- In both cases one needs the intersection of a left ideal L with a principal subalgebra $\mathbb{K}[g]$ for some polynomial g.

Computing Bernstein-Sato Polynomial: Examples

Name	Input	deg $b_f(s)$
uw_3	$xyz(x+z)(y-z)$	9
uw_4	$xyz(x+y+z)(3x+2y+z)$	8
uw_{14}	$xyz(2y+z)(y+z)(4y+z)(3y+z)$	11
uw_{18}	$(x-z)xyz(2x+2y-z)(y-z)(y+z)$	18
uw_{19}	$xyz(x-y-2z)(x+y+2z)(y-z)(y+z)$	18
ab23	$(z^2+w^3)(2zx+3w^2y)$	13
cnu7	$(xz+y)(x^7-y^7)$	10
reiffen11	$x^{11}+y^{11}+xy^{10}$	20
tt43	$x^4+y^4+z^4-(xyz)^3$	8
xyzcusp45	$(xy+z)(y^4+z^5+yz^4)$	20

Computing Bernstein-Sato Polynomial: Timings

Ex	ASIR		M2	SINGULAR		
	`bfct`	`bfunction`		`bfct`	`bfct` (+heuristic)	`bfctAnn`
uw_3	0:04	0:02	0:19	0:03	0:02	**0:01**
uw_4	1h:11m	**0:02**	17:56	0:03	0:04	**0:02**
uw_{14}	0:04	0:07	0:03	0:26	0:14	**0:01**
uw_{18}	$29h^{\times}$	7:26	$4h^{\times}$	9:48	**6:23**	12:24
uw_{19}	$24h^{\times}$	**1:16**	$3h^{\times}$	1:56	1:31	10:10
ab23	0:17	0:24	0:27	0:17	0:18	**0:04**
cnu7	4:46	7:31	$4h^{\times}$	23:09	**0:06**	0:19
reif11	0:48	0:11	0:34	0:10	0:16	**0:03**
tt43	0:05	0:07	0:05	0:17	0:17	**0:01**
xyz45	**1:09**	1:53	4h:18m	45:41	3:08	2:52

Important Conclusions

By systematically testing our implementations on the big set of nontrivial examples we observe, that

- hyperplane arrangements are generically best computed with initial-based methods
- hypersurfaces with non-isolated singularities (which are not h. a.'s) are generically best achieved with Ann f^s based methods

Hence, the important conclusion follows:

no algorithm is clearly superior over others!

Instead, there are classes, when one performs distinctly better than the others.

Open Question: determine more classes and work on heuristics.

Bernstein-Sato Polynomial of an Affine Variety

After the paper of Budur, Mustaţă and Saito, we formulated and proved the algorithm to compute the Bernstein-Sato polynomial for an affine variety.

Theorem

For every r-tuple $f = (f_1, \ldots, f_r) \in \mathbb{K}[\mathbf{x}]^r$ *there exists a non-zero univariate polynomial* $b(s) \in \mathbb{K}[s]$ *and r differential operators* $P_1(S), \ldots, P_r(S) \in D_n\langle S\rangle$ *such that*

$$\sum_{k=1}^{r} P_k(S) f_k \bullet f^s = b(s_1 + \cdots + s_r) \cdot f^s \qquad (1)$$

Hence, we are again interested in the generalized Bernstein data in this new setting. This is clearly more involved than the special case of a hypersurface.

Annihilator in the case of variety

Let $f = (f_1, \ldots, f_r)$ be an r-tuple in $\mathbb{K}[x]^r$. Moreover, let $s = (s_1, \ldots, s_r)$, $\frac{1}{f} = \frac{1}{f_1 \cdots f_r}$ and $f^s = f_1^{s_1} \cdots f_r^{s_r}$.

Consider $M = \mathbb{K}[x, s, \frac{1}{f}] \cdot f^s$, a free $\mathbb{K}[x, s, \frac{1}{f}]$-module of rank one generated by the formal symbol f^s.

Denote by $\mathbb{K}\langle S\rangle$ the universal enveloping algebra $U(\mathfrak{gl}_r)$, generated by the set of variables $S = (s_{ij})$, $i, j = 1, \ldots, r$, subject to relations:

$$s_{ij}s_{kl} - s_{kl}s_{ij} = \delta_{jk}s_{il} - \delta_{il}s_{kj}.$$

Then $D_n\langle S\rangle := D_n \otimes_{\mathbb{K}} \mathbb{K}\langle S\rangle$ is a G-algebra of Lie type.

The module M has a natural structure of a left $D_n\langle S\rangle$-module when the variables s_{ij} act as follows ($i \le j, g(s) \in \mathbb{K}[x, s, \frac{1}{f}]$):

$$s_{ij} \bullet (g(s) \cdot f^s) = s_i \cdot g(s + \epsilon_j - \epsilon_i) \frac{f_j}{f_i} \cdot f^s \in M,$$

Let $T = K[t], t = (t_1, \ldots, t_r)$ and $\partial_t = (\partial t_1, \ldots, \partial t_r)$. Then M is a $D_n(R) \otimes_{\mathbb{K}} D_r(T)$-module via

$$\begin{aligned} t_i \bullet (g(s) \cdot f^s) &= g(s + \epsilon_i) f_j \cdot f^s, \\ \partial t_i \bullet (g(s) \cdot f^s) &= -s_i g(s - \epsilon_i) \tfrac{1}{f_i} \cdot f^s. \end{aligned}$$

Note: the action of s_{ij} corresponds to the action of $-\partial t_i \cdot t_j$.

Consider the generalized Malgrange's ideal $I_f \subset D_n(R) \otimes_{\mathbb{K}} D_r(T)$

$$I_f = \langle t_i - f_i\, ,\ \partial_m + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_m} \partial t_j \mid 1 \le i \le r, 1 \le m \le n \rangle.$$

The module M has a natural structure of a left $D_n\langle S\rangle$-module when the variables s_{ij} act as follows ($i \leq j, g(s) \in \mathbb{K}[x, s, \frac{1}{f}]$):

$$s_{ij} \bullet (g(s) \cdot f^s) = s_i \cdot g(s + \epsilon_j - \epsilon_i)\, \frac{f_j}{f_i} \cdot f^s \; \in \; M,$$

Let $T = K[t], t = (t_1, \ldots, t_r)$ and $\partial_t = (\partial t_1, \ldots, \partial t_r)$. Then M is a $D_n(R) \otimes_{\mathbb{K}} D_r(T)$-module via

$$\begin{aligned} t_i \bullet (g(s) \cdot f^s) &= g(s + \epsilon_i) f_j \cdot f^s, \\ \partial t_i \bullet (g(s) \cdot f^s) &= -s_i g(s - \epsilon_i) \tfrac{1}{f_i} \cdot f^s. \end{aligned}$$

Note: the action of s_{ij} corresponds to the action of $-\partial t_i \cdot t_j$.

Consider the generalized Malgrange's ideal $I_f \subset D_n(R) \otimes_{\mathbb{K}} D_r(T)$

$$I_f = \langle t_i - f_i\,,\; \partial_m + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_m} \partial t_j \mid 1 \leq i \leq r, 1 \leq m \leq n \rangle.$$

The module M has a natural structure of a left $D_n\langle S\rangle$-module when the variables s_{ij} act as follows ($i \leq j, g(s) \in \mathbb{K}[x, s, \frac{1}{f}]$):

$$s_{ij} \bullet (g(s) \cdot f^s) = s_i \cdot g(s + \epsilon_j - \epsilon_i) \frac{f_j}{f_i} \cdot f^s \;\in\; M,$$

Let $T = K[t], t = (t_1, \ldots, t_r)$ and $\partial_t = (\partial t_1, \ldots, \partial t_r)$. Then M is a $D_n(R) \otimes_{\mathbb{K}} D_r(T)$-module via

$$\begin{array}{rcl} t_i \bullet (g(s) \cdot f^s) & = & g(s + \epsilon_i) f_j \cdot f^s, \\ \partial t_i \bullet (g(s) \cdot f^s) & = & -s_i g(s - \epsilon_i) \frac{1}{f_i} \cdot f^s. \end{array}$$

Note: the action of s_{ij} corresponds to the action of $-\partial t_i \cdot t_j$.

Consider the generalized Malgrange's ideal $I_f \subset D_n(R) \otimes_{\mathbb{K}} D_r(T)$

$$I_f = \langle t_i - f_i\,,\; \partial_m + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_m} \partial t_j \mid 1 \leq i \leq r, 1 \leq m \leq n \rangle.$$

Lemma

$\mathrm{Ann}_{D_n(R)\otimes_{\mathbb{K}} D_r(T)}(f^s) = I_f.$

Theorem

Let $f = (f_1, \ldots, f_r)$ be an r-tuple in $\mathbb{K}[x]^r$ and let $D_n\langle\partial_t, S\rangle$ be the $\mathbb{K}$-algebra generated by ∂_t and S over D_n subject to the corresponding non-commutative relations. Consider the left ideal in $D_n\langle\partial_t, S\rangle$

$$F := \left\langle s_{ij} + \partial t_i f_j \, , \; \partial_m + \sum_{k=1}^{r} \frac{\partial f_k}{\partial x_m} \partial t_k \;\middle|\; \begin{array}{l} 1 \le i,j \le r \\ 1 \le m \le n \end{array} \right\rangle.$$

Then $\mathrm{Ann}_{D\langle S\rangle}(f^s) = D_n\langle\partial_t, S\rangle F \cap D_n\langle S\rangle$.

Lemma

$\mathrm{Ann}_{D_n(R)\otimes_{\mathbb{K}} D_r(T)}(f^s) = I_f.$

Theorem

Let $f = (f_1, \ldots, f_r)$ be an r-tuple in $\mathbb{K}[x]^r$ and let $D_n\langle\partial_t, S\rangle$ be the $\mathbb{K}$-algebra generated by ∂_t and S over D_n subject to the corresponding non-commutative relations. Consider the left ideal in $D_n\langle\partial_t, S\rangle$

$$F := \left\langle s_{ij} + \partial t_i f_j \,,\ \partial_m + \sum_{k=1}^{r} \frac{\partial f_k}{\partial x_m} \partial t_k \;\middle|\; \begin{array}{c} 1 \le i, j \le r \\ 1 \le m \le n \end{array} \right\rangle.$$

Then $\mathrm{Ann}_{D\langle S\rangle}(f^s) = D_n\langle\partial_t, S\rangle F \cap D_n\langle S\rangle$.

Thank you for your attention!

Please visit the SINGULAR homepage

- http://www.singular.uni-kl.de/