Principal Intersection and Bernstein-Sato Polynomial of an Affine Variety

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**Introduction and notations**

**Basic notations**

- $\mathbb{K}$ stands for a field of characteristic 0 ($\mathbb{K} = \mathbb{C}$).
- $\mathbb{K}[s]$ the ring of polynomials in one variable over $\mathbb{C}$.
- $\mathbb{K}[x_1, \ldots, x_n]$ the ring of polynomials in $n$ variables.

$$D_n := \mathbb{K}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

the ring of $\mathbb{K}$-linear differential operators on $\mathbb{K}[x_1, \ldots, x_n]$, the $n$-th Weyl algebra:

$$\partial_i x_i = x_i \partial_i + 1, \text{ while for } i \neq j \partial_i x_j = x_j \partial_i, \partial_j \partial_i = \partial_i \partial_j, x_j x_i = x_i x_j$$

- $D_n[s] := D_n \otimes_{\mathbb{K}} \mathbb{K}[s]$. 

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The $D_n[s]$-module $\mathbb{K}[x, s, \frac{1}{f}] \cdot f^s$

- Let $f \in \mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ be a non-zero polynomial.
- By $\mathbb{K}[x, s, \frac{1}{f}]$ we denote the ring of rational functions of the form
  \[ \frac{g(x, s)}{f^r} \]
  where $g(x, s) \in \mathbb{K}[x, s] = \mathbb{K}[x_1, \ldots, x_n, s]$.
- We denote by $\mathbb{K}[x, s, \frac{1}{f}] \cdot f^s$ the free $\mathbb{K}[x, s, \frac{1}{f}]$-module of rank one generated by the formal symbol $f^s$.
- $\mathbb{K}[x, s, \frac{1}{f}] \cdot f^s$ has a natural structure of left $D_n[s]$-module.

\[ \partial_i \cdot f^s = s \frac{\partial f}{\partial x_i} \frac{1}{f} \cdot f^s \in \mathbb{K}[x, s, \frac{1}{f}] \cdot f^s \]
The global $b$-function a.k.a. Bernstein-Sato polynomial

**Theorem (J. Bernstein, 1972)**

For a non-constant polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ there exists a linear partial differential operator $P(s)$ with polynomial coefficients in $\mathbb{C}[x_1, \ldots, x_n, s]$ and a univariate polynomial $b(s) \in \mathbb{C}[s]$, such that

$$P(s) \cdot f^{s+1} = b(s) \cdot f^s,$$

**Definition (Bernstein & Sato)**

The set of all possible polynomials $b(s)$ satisfying the above equation is an ideal of $\mathbb{C}[s]$. The monic generator of this ideal is denoted by $b_f(s)$ and called the Bernstein-Sato polynomial of $f$.

$$\text{Ann}_{D[s]}(f^s) = \{ Q(s) \in D[s] \mid Q(s) \cdot f^s = 0 \} \subset D[s]$$
Reformulation of Bernstein’s Theorem

Given \( f \in K[x] \) a polynomial, there exists an operator \( P(s) \in D[s] \) and \( b_f(s) \in K[s] \), such that

\[
P(s) \cdot f^{s+1} = b_f(s) \cdot f^s \iff P(s)f - b_f(s) \in \text{Ann}_{D[s]} f^s.
\]

The monic polynomial \( b_f(s) \) of smallest degree, satisfying the above condition is the Bernstein-Sato polynomial.

Theorem (Kashiwara)

All roots of \( b_f(s) \) are negative rational numbers.

Algorithm for computing Bernstein-Sato polynomial via \( \text{Ann}_{D[s]} f^s \)

\[
(\text{Ann}_{D[s]} f^s + \langle f \rangle) \cap K[s] = \langle b(s) \rangle
\]
Evolution from OT to BM

1. **Oaku and Takayama** proved in 1999, that \( \text{Ann}_{D[s]}(f^s) = \)

\[
D\langle t, \partial_t \mid [\partial_t, t] = 1 \rangle[u, v](\langle t - uf, uv - 1, \{u \frac{\partial f}{\partial x_i}\partial_t + \partial_i \} \rangle) \cap D[-t\partial_t - 1],
\]

"eliminate \( u, v \) and intersect the result with \( D[s], s = -t\partial_t - 1 \)."

2. **Briançon and Maisonobe** proved in 2002, that \( \text{Ann}_{D[s]}(f^s) = \)

\[
D\langle \partial_t, s \mid [s, \partial_t] = \partial_t \rangle\langle f\partial_t + s, \{\frac{\partial f}{\partial x_i}\partial_t + \partial_i \} \rangle \cap D[s]
\]

"eliminate \( \partial_t \)."

3. F. Castro and J.-M. Ucha, JSC 2004 proved, that

*The BM method is less complex than the method of OT.*

4. ISSAC 09: we give a new proof for the BM method, showing that

*The BM is a consequence of the OT method.*
New Proof for Briançon and Maisonobe’s Method

Ingredients

- **G-algebras of Lie type** (i.e. with relations $\forall 1 \leq i < j \leq n$ $x_jx_i = x_ix_j + d_{ij}$, $d_{ij} \in K[x]$)


- preimage of a left ideal under a morphism of $K$-algebras (V. Levandovskyy, ISSAC 2006)

- slim Gröbner basis algorithm (M. Brickenstein, 2005)
Adapted Preimage Theorem

Assume that $A = \mathbb{K}\langle x \rangle / I_A$ and $B = \mathbb{K}\langle y \rangle / I_B$ be $G$-algebras of Lie type and $\phi : A \to B$ be a homomorphism of $\mathbb{K}$-algebras. Then

$A \otimes^\phi_k B := \mathbb{K}\langle x, y \rangle / (I_A + I_B + \langle \{ y_j x_i - x_i y_j - y_j \phi(x_i) + \phi(x_i) y_j \} \rangle)$.

Suppose, that there exists an elimination ordering for $B$ on $A \otimes^\phi_k B$, satisfying $\text{lm}(y_j \phi(x_i) - \phi(x_i) y_j) \prec x_i y_j$ for all $1 \leq i \leq n, 1 \leq j \leq m$.

Let $A_1, B_1, C$ be $G$-algebras of Lie type and $\varphi : A_1 \to B_1$ be a homomorphism of $\mathbb{K}$-algebras. Consider the following data:

\[
A = C \otimes_k A_1, \quad B = C \otimes_k B_1, \quad \phi = 1_C \otimes \varphi : A \to B,
E = A \otimes^\phi_k B, \quad E' = C \otimes_k (A_1 \otimes^\varphi_k B_1).
\]

Then $E', A \otimes^\phi_k B, E$ are $G$-alg of Lie type, and for a left ideal $J \subset B$

1. $(E I_\phi + E J) \cap E' = E' I_\phi + E' J$.
2. $\phi^{-1}(J) = (E' I_\phi + E' J) \cap A \subset (E' \cap A)$.

The latter intersection can be computed using Gröbner bases.
Theorem (Briançon and Maisonobe)

Let $S_p = \mathbb{K}\langle \partial t_j, s_j \mid \partial t_j s_k = s_k \partial t_j - \delta_{jk} \partial t_j \rangle$ (the p-th shift algebra) and $B = D_n \otimes_{\mathbb{K}} S_p$. Moreover, consider the left ideal $I \subset B$, generated by

$$\{ s_j + f_j \partial t_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k \}.$$  

Then $\text{Ann}_{D_n[s]}(f^s) = I \cap D_n[s]$.

Setup

Let $A := D_n[s] = \mathbb{K}\langle \{ s_j, x_i, \partial_i \} \mid \partial_i x_i = x_i \partial_i + 1 \rangle$, and $B := \mathbb{K}\langle \{ t_j, \partial t_j, x_i, \partial_i \} \mid \{ \partial_i x_i = x_i \partial_i + 1, \partial t_j t_j = t_j \partial t_j + 1 \} \rangle$.

Thus in the notations of Preimage Theorem $C = D_n, A_1 = \mathbb{K}[s], B_1 = \mathbb{K}\langle \{ t_j, \partial t_j \mid \partial t_j t_j = t_j \partial t_j + 1 \} \rangle$ and $A = C \otimes A_1, B = C \otimes B_1$.

Consider $\varphi : A_1 \to B_1, \ s_j \mapsto -t_j \partial t_j - 1$, then $I_\varphi = \langle \{ t_j \partial t_j + s_j + 1 \} \rangle \subset A_1 \otimes_{\mathbb{K}} \varphi B_1 =: E'$.
**Theorem (Briançon and Maisonobe)**

Let $S_p = \mathbb{K}\langle\{\partial t_j, s_j\} \mid \partial t_j s_k = s_k \partial t_j - \delta_{jk} \partial t_j \rangle$ (the $p$-th shift algebra) and $B = D_n \otimes_{\mathbb{K}} S_p$. Moreover, consider the left ideal $I \subset B$, generated by

$\{s_j + f_j \partial t_j, \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k\}$. Then $\text{Ann}_{D_n[s]}(f^s) = I \cap D_n[s]$.

**Setup**

Let $A := D_n[s] = \mathbb{K}\langle\{s_j, x_i, \partial_i\} \mid \partial_i x_i = x_i \partial_i + 1\rangle$, and $B := \mathbb{K}\langle\{t_j, \partial t_j, x_i, \partial_i\} \mid \{\partial_i x_i = x_i \partial_i + 1, \partial t_j t_j = t_j \partial t_j + 1\}\rangle$. Thus in the notations of Preimage Theorem

$C = D_n, A_1 = \mathbb{K}[s], B_1 = \mathbb{K}\langle\{t_j, \partial t_j \mid \partial t_j t_j = t_j \partial t_j + 1\}\rangle$ and $A = C \otimes A_1, B = C \otimes B_1$.

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$I_\varphi = \langle\{t_j \partial t_j + s_j + 1\}\rangle \subset A_1 \otimes_{\mathbb{K}}^\varphi B_1 =: E'$. 
Since \([t_k, s_j] = \delta_{jk} t_j\) and \([\partial t_k, s_j] = -\delta_{jk} \partial t_j\), the ordering conditions of Preimage Theorem are \(t_j \prec s_j t_j, \partial t_j \prec s_j \partial t_j\). They are satisfied iff \(1 \leq t_j, \partial t_j, s_j\).

By Preimage Theorem, for any \(L \subset B\), \(\phi^{-1}(L) = (I_\phi + L) \cap A\). We apply it to the Malgrange ideal

\[
L = \langle\{t_j - f_j, \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j\}\rangle.
\]

Since \(t_j \partial t_j + s_j + 1\) reduces to \(f_j \partial t_j + s_j \in I_\phi + L\), we have

\[
I_\phi + L = \langle\{t_j - f_j, \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial t_j, f_j \partial t_j + s_j\}\rangle.
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\]
Lemma

Consider an ordering $\prec_T$, which satisfies the property $\{t_j\} \succ \{x_i\}$, $\{\partial_i, s_j\} \succ \{x_i, \partial t_j\}$. Moreover, set

$$g_i := \partial_i + \sum_{k=1}^{p} \frac{\partial f_k}{\partial x_i} \partial t_k, \quad S_1 := \{t_j - f_j, g_i\},$$

$$S_2 := S_1 \cup \{s_j + f_j \partial t_j\} \subset E'.$$

Then $S_1$ and $S_2$ are left Gröbner bases with respect to $\prec_T$.

Proof: numerous applications of generalized product criterion.

Final Step

Since an ordering, eliminating $\{t_j\}$, satisfies the property in the Lemma, it remains to eliminate $\{\partial t_j\}$ from $S_2 \setminus \{t_i - f_i\}$, what is exactly the statement of the Theorem of Briançon and Maisonobe.
Computer algebra systems for D-modules

Computer algebra system **SINGULAR** has joined the club of systems, able to perform computations for $D$-module theory. To the best of our knowledge, this club consists of

- experimental program **kan/sm1** by N. Takayama
- the package **Dmodules** in Macaulay2 by A. Leykin and H. Tsai
- **bfct** package in Risa/Asir by M. Noro
- there are rumours about ongoing development in **CoCoA**

The perfect package for $D$-modules must combine flexibility and rich functionality with high performance, e.g. in order to treat complicated examples, coming both from theory and applications.
Initial ideal with respect to weights

**Definition**

Let \( 0 \neq w \in \mathbb{R}^n_{\geq 0} \). For a non-zero element of \( D \), there exists \( m \in \mathbb{R} \) such that

\[
p = \sum_{\alpha, \beta \in \mathbb{N}_0^n: -w\alpha + w\beta = m} c_{\alpha\beta} x^{\alpha} \partial^{\beta} + \sum_{\alpha, \beta \in \mathbb{N}_0^n: -w\alpha + w\beta < m} c_{\alpha\beta} x^{\alpha} \partial^{\beta}
\]

We call the graded ideal \( \text{in}_{(-w,w)}(I) := \mathbb{K} \cdot \{ \text{in}_{(-w,w)}(p) \mid p \in I \} \) the *initial ideal of \( I \) with respect to the weight \( w \).*

**Definition**

Let \( 0 \neq w \in \mathbb{R}^n_{\geq 0} \) and \( s := \sum_{i=1}^n w_i x_i \partial_i \). Then \( \text{in}_{(-w,w)}(I) \cap \mathbb{K}[s] \) is a principal ideal in \( \mathbb{K}[s] \). Its monic generator \( b(s) \) is called the *global \( b \)-function of \( I \) with respect to the weight \( w \).*
Principal Intersection: Problem Formulation

\[ A \quad \text{associative } K\text{-algebra} \]
\[ J \subset A \quad \text{left ideal} \]
\[ G \quad \text{finite Gröbner basis of } J \text{ wrt. arbitrary global ordering} \]
\[ s \in A \setminus K \quad \text{any nonconstant element} \]

What can we say a priori about \( J \cap K[s] \)?

There are the following situations:

1. \( \text{Im}(g) \nmid \text{Im}(s^k) \) for all \( g \in G, k \in \mathbb{N}_0 \)
2. Situation 1 does not hold, that is \( \exists g \in G, k \geq 1 \text{ with } \text{Im}(g) \mid \text{Im}(s^k) \)

1. \( J \cdot s \subset J \) and \( \dim_K (\text{End}_A(A/J)) < \infty \)
2. Situation 2.1 does not hold: 
   1. ????
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\( A \) associative \( \mathbb{K} \)-algebra
\( J \subset A \) left ideal
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Situations 1 und 2

Lemma (Situation 1)

If there exists no \( g \in G \) such that \( \text{Im}(g) \) divides \( \text{Im}(s^k) \) for some \( k \in \mathbb{N}_0 \), then \( J \cap \mathbb{K}[s] = 0 \).

Situation 2

The converse to the lemma does not hold. Consider \( J = \langle y^2 + x \rangle \subseteq \mathbb{K}[x, y] \). Then \( J \cap \mathbb{K}[y] = \{0\} \) while \( \{y^2 + x\} \) is a Gröbner basis of \( J \) for any ordering.
Lemma (Situation 1)

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Lemma

Let $J \cdot s \subset J$ and $\text{End}_A(A/J)$ finite dimensional as $K$-vector space. Then $J \cap K[s] \neq 0$.

Proof:

- $\cdot s : A/J \rightarrow A/J$, $[a] \mapsto [a \cdot s]$ is well-defined $A$-module endomorphism
- $\cdot s$ has minimal polynomial $\mu$
- $\mu \in K[s] \cap J$ and $\mu \neq 0$ (even $\mu \notin K$)

The condition $\dim_K \text{End}_A(A/J) < \infty$ is fulfilled, if

- $A/J$ is finite dimensional over $K$ or
- $A$ is Weyl algebra and $A/J$ is a holonomic $A$-module.
Situation 2.1

**Lemma**

Let $J \cdot s \subset J$ and $\text{End}_A(A/J)$ finite dimensional as $K$-vector space. Then $J \cap K[s] \neq 0$.

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- $A$ is Weyl algebra and $A/J$ is a holonomic $A$-module.
Algorithm (pIntersect)

Input:  $s \in A \setminus K$, $J \subset A$ such that $J \cap K[s] \neq \{0\}$
Output: $b \in K[s]$ normed such, that $J \cap K[s] = \langle b \rangle$

$G := \text{finite Gröbner basis of } J \text{ wrt. arbitrary global ordering}$
$i := 1$
loop
  if there exist $a_0, \ldots, a_{i-1} \in K$ with

  \[ \text{NF}(s^i, G) + \sum_{j=0}^{i-1} a_j \text{NF}(s^j, G) = 0 \]  

  then
    return $b := s^i + \sum_{j=0}^{i-1} a_j s^j$
  else
    $i := i + 1$
  end if
end loop
**Theorem**

For a holonomic module $D/I$, a $b$-function of $I$ wrt $(-w, w)$ is not constant.

**Lemma**

Let $A$ be a $\mathbb{K}$-algebra, $J \subset A$ a left ideal and let $s \in A$. For $i \in \mathbb{N}$ put $r_i = \text{NF}(s^i, J)$ and $q_i = s^i - r_i \in J$. Then we have for all $i \in \mathbb{N}$

$$r_{i+1} = \text{NF}([s^i - r_i, r_1] + r_ir_1, J).$$

Note, that computing Lie bracket $[f, g]$ both in theory and in practice is easier and faster, than to compute $[f, g]$ as $f \cdot g - g \cdot f$. 
Consequence and Improvement

**Theorem**

For a holonomic module $D/I$, a b-function of $I$ wrt $(-w, w)$ is not constant.

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Note, that computing Lie bracket $[f, g]$ both in theory and in practice is easier and faster, than to compute $[f, g]$ as $f \cdot g - g \cdot f$. 
Algorithm (intersectUpTo)

Input: $s_1, \ldots, s_r \in A \setminus \mathbb{K}, s_j s_i = s_i s_j, J \subset A$ left ideal, $k \in \mathbb{N}$

Output: Gröbner basis $B$ of $J \cap \mathbb{K}[s_1, \ldots, s_r]$ up to degree $k$

$G :=$ Gröbner basis of $J$ wrt. arbitrary global ordering

$d := 0, \quad B := \emptyset$

while $d \leq k$ do

$M_d := \{s^\alpha \mid |\alpha| \leq d\}$

if $\exists a_m \in \mathbb{K}$ with $\sum_{m \in M_d} a_m \text{NF}(m, G) = 0$ then

$f := \sum_{m \in M_d} a_m m$

if $f \neq 0$ and $f \notin \langle B \rangle$ then

$B := B \cup \{f\}$

end if

end if

$d := d + 1$

end while

return $B$
Fine tuning and Termination

- Found $p \in B$ with $\text{Im}(p) = m$, ignore all multiples of $m$
- If $G$ is a Gröbner basis of $J$ wrt. $\prec$, ignore

$$\{ m \in M_d \mid \max(m' \in L(G) \cap M_d) \prec m \},$$

since $p \in J \cap \mathbb{K}[s_1, \ldots, s_r] \iff \text{Im}(p) \in L(G) \cap \mathbb{K}[s_1, \ldots, s_r]$.
- $\text{NF}(m, G) = m$, if $m \notin L(G) \cap \mathbb{K}[s_1, \ldots, s_r]$.

In general, the termination is provided in case when
- $J \cap \mathbb{K}[s_1, \ldots, s_r]$ is zero-dimensional or
- it is known in advance, that $J \cap \mathbb{K}[s_1, \ldots, s_r]$ is principal.
Computing Bernstein Data

- There are several methods to compute $\text{Ann}_{D[s]} f^s \subset D[s]$ (BM etc.)
- For computing $b(s) \in K[s]$, one uses e.g.
  
  $\langle b(s) \rangle = (\text{Ann}_{D[s]} f^s + \langle f \rangle) \cap K[s]$. Note, that this computation via elimination ranges from "very hard" to "hopeless" in practice.

$\langle b(s) \rangle = \text{in}_{(-w,w)}(\langle t - f, \{ \frac{\partial f}{\partial x_i} \partial_t + \partial_i \} \rangle) \cap K[\theta]$ for $w \in \mathbb{R}^{n+1}$ and

$\theta = w_0 t \partial_t + \sum_{i=1}^n w_i x_i \partial_i$.

- In both cases one needs the intersection of a left ideal $L$ with a principal subalgebra $K[g]$ for some polynomial $g$. 
## Computing Bernstein-Sato Polynomial: Examples

<table>
<thead>
<tr>
<th>Name</th>
<th>Input</th>
<th>deg $b_f(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>uw$_3$</td>
<td>$xyz(x + z)(y - z)$</td>
<td>9</td>
</tr>
<tr>
<td>uw$_4$</td>
<td>$xyz(x + y + z)(3x + 2y + z)$</td>
<td>8</td>
</tr>
<tr>
<td>uw$_{14}$</td>
<td>$xyz(2y + z)(y + z)(4y + z)(3y + z)$</td>
<td>11</td>
</tr>
<tr>
<td>uw$_{18}$</td>
<td>$(x - z)xyz(2x + 2y - z)(y - z)(y + z)$</td>
<td>18</td>
</tr>
<tr>
<td>uw$_{19}$</td>
<td>$xyz(x - y - 2z)(x + y + 2z)(y - z)(y + z)$</td>
<td>18</td>
</tr>
<tr>
<td>ab$_{23}$</td>
<td>$(z^2 + w^3)(2zx + 3w^2y)$</td>
<td>13</td>
</tr>
<tr>
<td>cnu$_7$</td>
<td>$(xz + y)(x^7 - y^7)$</td>
<td>10</td>
</tr>
<tr>
<td>reiffen$_{11}$</td>
<td>$x^{11} + y^{11} + xy^{10}$</td>
<td>20</td>
</tr>
<tr>
<td>tt$_{43}$</td>
<td>$x^4 + y^4 + z^4 - (xyz)^3$</td>
<td>8</td>
</tr>
<tr>
<td>xyzcusp$_{45}$</td>
<td>$(xy + z)(y^4 + z^5 + yz^4)$</td>
<td>20</td>
</tr>
</tbody>
</table>
## Computing Bernstein-Sato Polynomial: Timings

<table>
<thead>
<tr>
<th>Ex</th>
<th>ASIR</th>
<th>M2</th>
<th>SINGULAR</th>
</tr>
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<tr>
<td></td>
<td>bfct</td>
<td>bfunction</td>
<td>bfct</td>
</tr>
<tr>
<td>uw3</td>
<td>0:04</td>
<td>0:02</td>
<td>0:19</td>
</tr>
<tr>
<td>uw4</td>
<td>1h:11m</td>
<td>0:02</td>
<td>17:56</td>
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<tr>
<td>uw14</td>
<td>0:04</td>
<td>0:07</td>
<td>0:03</td>
</tr>
<tr>
<td>uw18</td>
<td>29h×</td>
<td>7:26</td>
<td>4h×</td>
</tr>
<tr>
<td>uw19</td>
<td>24h×</td>
<td>1:16</td>
<td>3h×</td>
</tr>
<tr>
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<td>0:17</td>
<td>0:24</td>
<td>0:27</td>
</tr>
<tr>
<td>cnu7</td>
<td>4:46</td>
<td>7:31</td>
<td>4h×</td>
</tr>
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<td>0:11</td>
<td>0:34</td>
</tr>
<tr>
<td>tt43</td>
<td>0:05</td>
<td>0:07</td>
<td>0:05</td>
</tr>
<tr>
<td>xyz45</td>
<td>1:09</td>
<td>1:53</td>
<td>4h:18m</td>
</tr>
</tbody>
</table>
Important Conclusions

By systematically testing our implementations on the big set of nontrivial examples we observe, that

- hyperplane arrangements are generically best computed with initial-based methods
- hypersurfaces with non-isolated singularities (which are not h. a.’s) are generically best achieved with Ann $f^s$ based methods

Hence, the important conclusion follows:

no algorithm is clearly superior over others!

Instead, there are classes, when one performs distinctly better than the others.

Open Question: determine more classes and work on heuristics.
Bernstein-Sato Polynomial of an Affine Variety

After the paper of Budur, Mustaţă and Saito, we formulated and proved the algorithm to compute the Bernstein-Sato polynomial for an affine variety.

**Theorem**

For every $r$-tuple $f = (f_1, \ldots, f_r) \in K[x]^r$ there exists a non-zero univariate polynomial $b(s) \in K[s]$ and $r$ differential operators $P_1(S), \ldots, P_r(S) \in D_n\langle S \rangle$ such that

$$
\sum_{k=1}^{r} P_k(S) f_k \cdot f^s = b(s_1 + \cdots + s_r) \cdot f^s
$$

Hence, we are again interested in the generalized Bernstein data in this new setting. This is clearly more involved than the special case of a hypersurface.
Annihilator in the case of variety

Let \( f = (f_1, \ldots, f_r) \) be an \( r \)-tuple in \( \mathbb{K}[x]^r \). Moreover, let \( s = (s_1, \ldots, s_r) \),

\[
\frac{1}{f} = \frac{1}{f_1 \cdots f_r}
\]

and \( f^s = f_1^{s_1} \cdots f_r^{s_r} \).

Consider \( M = \mathbb{K}[x, s, \frac{1}{f}] \cdot f^s \), a free \( \mathbb{K}[x, s, \frac{1}{f}] \)-module of rank one generated by the formal symbol \( f^s \).

Denote by \( \mathbb{K}\langle S \rangle \) the universal enveloping algebra \( U(\mathfrak{gl}_r) \), generated by the set of variables \( S = (s_{ij}), \ i, j = 1, \ldots, r \), subject to relations:

\[
s_{ij} s_{kl} - s_{kl} s_{ij} = \delta_{jk} s_{il} - \delta_{il} s_{kj}.
\]

Then \( D_n\langle S \rangle := D_n \otimes_{\mathbb{K}} \mathbb{K}\langle S \rangle \) is a \( G \)-algebra of Lie type.
The module $M$ has a natural structure of a left $D_n \langle S \rangle$-module when the variables $s_{ij}$ act as follows ($i \leq j, g(s) \in \mathbb{K}[x, s, \frac{1}{t}]$):

$$s_{ij} \cdot (g(s) \cdot f^s) = s_i \cdot g(s + \epsilon_j - \epsilon_i) \frac{f_j}{f_i} \cdot f^s \in M,$$

Let $T = K[t], t = (t_1, \ldots, t_r)$ and $\partial_t = (\partial t_1, \ldots, \partial t_r)$. Then $M$ is a $D_n(R) \otimes_{\mathbb{K}} D_r(T)$-module via

$$t_i \cdot (g(s) \cdot f^s) = g(s + \epsilon_i)f_j \cdot f^s,$$

$$\partial t_i \cdot (g(s) \cdot f^s) = -s_i g(s - \epsilon_i)\frac{1}{f_i} \cdot f^s.$$

Note: the action of $s_{ij}$ corresponds to the action of $-\partial t_i \cdot t_j$.

Consider the generalized Malgrange’s ideal $I_f \subset D_n(R) \otimes_{\mathbb{K}} D_r(T)$

$$I_f = \langle t_i - f_i, \partial_m + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_m} \partial t_j \mid 1 \leq i \leq r, 1 \leq m \leq n \rangle.$$
The module $M$ has a natural structure of a left $D_n\langle S \rangle$-module when the variables $s_{ij}$ act as follows ($i \leq j, g(s) \in \mathbb{K}[x, s, \frac{1}{t}]$):

$$s_{ij} \cdot (g(s) \cdot f^s) = s_i \cdot g(s + \epsilon_j - \epsilon_i) \frac{f_j}{f_i} \cdot f^s \in M,$$

Let $T = K[t], t = (t_1, \ldots, t_r)$ and $\partial_t = (\partial t_1, \ldots, \partial t_r)$. Then $M$ is a $D_n(R) \otimes_{\mathbb{K}} D_r(T)$-module via

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$$s_{ij} \bullet (g(s) \cdot f^s) = s_i \cdot g(s + \epsilon_j - \epsilon_i) \frac{f_j}{f_i} \cdot f^s \in M,$$

Let $T = K[t]$, $t = (t_1, \ldots, t_r)$ and $\partial_t = (\partial t_1, \ldots, \partial t_r)$. Then $M$ is a $D_n(R) \otimes_{\mathbb{K}} D_r(T)$-module via

$$t_i \bullet (g(s) \cdot f^s) = g(s + \epsilon_i)f_j \cdot f^s,$$
$$\partial t_i \bullet (g(s) \cdot f^s) = -s_i g(s - \epsilon_i)\frac{1}{f_i} \cdot f^s.$$

Note: the action of $s_{ij}$ corresponds to the action of $-\partial t_i \cdot t_j$.

Consider the generalized Malgrange’s ideal $I_f \subset D_n(R) \otimes_{\mathbb{K}} D_r(T)$

$$I_f = \langle t_i - f_i, \partial_m + \sum_{j=1}^{r} \frac{\partial f_j}{\partial x_m} \partial t_j \mid 1 \leq i \leq r, 1 \leq m \leq n \rangle.$$
Lemma

\[ \text{Ann}_{D_n(R) \otimes K D_r(T)}(f^S) = I_f. \]

Theorem

Let \( f = (f_1, \ldots, f_r) \) be an \( r \)-tuple in \( K[x]^r \) and let \( D_n\langle \partial_t, S \rangle \) be the \( K \)-algebra generated by \( \partial_t \) and \( S \) over \( D_n \) subject to the corresponding non-commutative relations. Consider the left ideal in \( D_n\langle \partial_t, S \rangle \)

\[ F := \langle s_{ij} + \partial_t f_j, \partial_m + \sum_{k=1}^{r} \frac{\partial f_k}{\partial x_m} \partial_{t_k} \mid 1 \leq i, j \leq r, 1 \leq m \leq n \rangle. \]

Then \( \text{Ann}_{D_\langle S \rangle}(f^S) = D_n\langle \partial_t, S \rangle F \cap D_n\langle S \rangle. \)
**Lemma**

\[ \text{Ann}_{D^n(R) \otimes_K D_r(T)}(f^S) = I_f. \]

**Theorem**

Let \( f = (f_1, \ldots, f_r) \) be an \( r \)-tuple in \( \mathbb{K}[x]^r \) and let \( D_n\langle \partial_t, S \rangle \) be the \( \mathbb{K} \)-algebra generated by \( \partial_t \) and \( S \) over \( D_n \) subject to the corresponding non-commutative relations. Consider the left ideal in \( D_n\langle \partial_t, S \rangle \)

\[
F := \left\langle s_{ij} + \partial_t f_j, \partial_m + \sum_{k=1}^{r} \frac{\partial f_k}{\partial x_m} \partial_{t_k} \bigg| \begin{array}{l} 1 \leq i, j \leq r \\ 1 \leq m \leq n \end{array} \right. \right. \]

Then \( \text{Ann}_{D\langle S \rangle}(f^S) = D_n\langle \partial_t, S \rangle F \cap D_n\langle S \rangle. \)
Thank you for your attention!

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