Solution of Algebraic Riccati Equations Using the Sum of Roots

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What We Want to Do....

Gröbner Basis — Algebraic method
Computational Algebraic Geometry

Sum of Roots

Polynomial Spectral Factorization
Important Mathematical Tool for Signal Processing and Control

Extend the result to solve Algebraic Riccati Equations!
Outline of the Talk

1. Polynomial Spectral Factorization via the Sum of Roots
2. Algebraic Riccati Equations
3. Sum of Roots Approach to Algebraic Riccati Equations
4. Concluding Remarks
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Polynomial Spectral Factorization

Problem Formulation

- **Given**: an even polynomial in $\lambda$ (polynomial in $\lambda^2$)

\[ f(\lambda) = \lambda^{2n} + a_{2n-2}\lambda^{2n-2} + a_{2n-4}\lambda^{2n-4} + \cdots + a_0 \]

- **Assumption**: $f(\lambda)$ has no roots on the imaginary axis

- **Roots of $f(\lambda)$**: symmetrical w.r.t. the real & imaginary axes and the origin

- **Task**: Find a polynomial $g(\lambda)$ such that

\[ f(\lambda) = (-1)^n g(\lambda)g(-\lambda) \]

and $g(\lambda)$ has roots in the open left half plane only.

- **$g(\lambda)$**: monic without loss of generality

\[ g(\lambda) = \lambda^n + \underbrace{\sigma \lambda^{n-1} + b_{n-2}\lambda^{n-2} + \cdots + b_0}_{\sum_{i=1}^{n} \alpha_i} \]

- **$\sigma$**: Sum of Roots — $\alpha_j$: Roots of $f(\lambda)$ in the open left half plane
Polynomial Spectral Factorization via the Sum of Roots

Solution Approach Using the Sum of Roots

\[ g(\lambda) = \lambda^n + \sigma \lambda^{n-1} + b_{n-2} \lambda^{n-2} + \cdots + b_0 \]

- \[ f(\lambda) = (-1)^n g(\lambda) g(-\lambda) \]
  \[ \Rightarrow \text{a system of algebraic equations in } \sigma, b_j \]
- Reduced Gröbner basis
- Efficient basis-conversion technique \[ \Rightarrow \text{Shape basis obtained} \]

**Shape Basis**

\[
\begin{align*}
S_f(\sigma) &= 0 & \text{— characteristic polynomial of } \sigma, 2^n-\text{th order} \\
\sigma_{n-2} &= h_{n-2}(\sigma) & \text{— true } \sigma = \text{largest real root of } S_f \\
\sigma_{n-3} &= h_{n-3}(\sigma) \\
\vdots & & h_i: \text{polynomial of order} \\
\sigma_0 &= h_0(\sigma) & \text{strictly less than } 2^n
\end{align*}
\]
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Algebraic Riccati Equations

Problem Formulation
A quadratic matrix equation for an unknown $n \times n$ matrix $X$ of the form

$$A^T X + XA + XRX + Q = 0$$

- $A, Q, R$ : given $n \times n$ matrices
- $Q, R$ : symmetric

Positive semi-definite $X$ ($X = X^T \geq 0$) to be found

Assumptions
- $Q$ positive semi-definite, $R$ negative semi-definite
- $(A, R)$ stabilizable, $(Q, A)$ detectable

$\implies$ Guarantees no roots on the imaginary axis
and the existence of positive semi-definite $X$
Algebraic Riccati Equations

**Target**

**Naïve Approach**

- Elements of $X = (x_{i,j})$: variables
- $A^T X + X A + X R X + Q = 0$
  $\implies$ A set of algebraic equations in $x_{i,j}$
- Employ general Gröbner basis computation algorithms

**Drawbacks**

- Cannot make use of structural properties
- # of variables: $\frac{1}{2}n(n + 1)$
- Difficult to characterize the positive semi-definite solution (?)

**Aim**

- Characterize the solution $X$ by means of the SoR
Algebraic Riccati Equations

Standard (Numerical) Solution Approach

\[ A^T X + X A + X R X + Q = 0 \]

1. Construct the Hamiltonian matrix

\[ H := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \quad (2n \times 2n \text{ matrix}) \]

2. Compute \( n \) eigenvectors \( v_i \)

associates with the eigenvalues of \( H \) in the left half plane.

3. \[
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \quad X_1, X_2 : n \times n \text{ matrices}
\]

4. \[ X = X_2 X_1^{-1} (\geq 0) \]
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Proposed Approach (1)

Input
- $A, Q, R \in \mathbb{Q}[q]^{n \times n}$
- $q = (q_1, \ldots, q_m)$: vector of parameters

Output
- $X$: rational functions of parameters $q$ and the SoR $\sigma$
- $S_f(\sigma)$: Characteristic polynomial relating parameters $q$ and the SoR $\sigma$

! Closed-form expression for $X$ in terms of parameters
— not sought / impossible
Proposed Approach (2)

Consists of 4 Steps....

1. Compute the characteristic polynomial $f(\lambda)$ of $H$. Carry out polynomial spectral factorization for $f(\lambda) = (-1)^n g(\lambda) g(-\lambda)$. — Relate parameters and the SoR

2. Express the eigenvector $v(\lambda')$ as polynomials in the eigenvalue $\lambda'$

3. Express $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ as a product of two matrices: $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} V$

   $\tilde{X}_1$, $\tilde{X}_2$: matrices in the coefficients of $g(\lambda)$ — SoR

   $V$: matrices in eigenvalues $\lambda_i$

   — Separate $\sigma$, $b_j$ and $\lambda_i$

4. $X = \tilde{X}_2 \tilde{X}_1^{-1}$
Step by Step (1)

\[ A = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} = \begin{bmatrix} -2 & -3 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -10 & -14 & 2 & -1 \\ -14 & -20 & 3 & 0 \end{bmatrix} \]

Characteristic polynomial \( f(\lambda) = \det (\lambda I_4 - H) = \lambda^4 - 28\lambda^2 + 35 \)

Polynomial spectral factorization for \( f(\lambda) \):

\[ g(\lambda) = \lambda^2 + \sigma \lambda + b_0 \]

\[ f(\lambda) = (-1)^n g(\lambda) g(-\lambda) \implies \begin{cases} \sigma^4 - 56\sigma^2 + 644 = 0 \\ b_0 = \frac{1}{2} \sigma^2 - 14 \end{cases} \]
Step by Step (2)

2. Get an expression for the eigenvector. Compute the quotient: 

\[ f(\lambda) = (\lambda - \lambda')q(\lambda) + r \]

\[ f(H) = (H - \lambda' I_{2n})q(H) = 0 \implies Hq(H)u = \lambda' q(H)u \]

for suitable chosen \( u \)

\[ q(\lambda) = \lambda^3 + \lambda' \lambda^2 + (\lambda'^2 - 28)\lambda + \lambda'^3 - 28\lambda' \]

\[ u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \implies v(\lambda') = q(H)u = \begin{bmatrix} 4\lambda' - 20 \\ -\lambda'^2 + 15 \\ -\lambda'^2 + 12\lambda' - 5 \\ \lambda'^3 - 11\lambda' - 20 \end{bmatrix} \]

\textit{The quotient to be computed only once!}
Step by Step (2)

2. Get an expression for the eigenvector.

Compute the quotient:

\[ f(\lambda) = (\lambda - \lambda')q(\lambda) + r \]

\[ f(H) = (H - \lambda' I_{2n})q(H) = 0 \quad \Rightarrow \quad H q(H)u = \lambda' q(H)u \]

for suitable chosen \( u \)

\[ q(\lambda) = \lambda^3 + \lambda' \lambda^2 + (\lambda'^2 - 28)\lambda + \lambda'^3 - 28\lambda' \]

\[ u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad v(\lambda') = q(H)u = \begin{bmatrix} 4\lambda' - 20 \\ -\lambda'^2 + 15 \\ -\lambda'^2 + 12\lambda' - 5 \\ \lambda'^3 - 11\lambda' - 20 \end{bmatrix} \]

The quotient to be computed only once!
Step by Step (3)

\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \begin{bmatrix}
v(\lambda_1) & v(\lambda_2) & \cdots & v(\lambda_n)
\end{bmatrix}
\]

— \(\lambda_i\) : eigenvalues in the left half plane

In fact, it suffices to consider \(v(\lambda_i) \mod g(\lambda_i)\)

\[
v(\lambda_i) \mod g(\lambda_i) = \begin{bmatrix}
4\lambda_i - 20 \\
\sigma \lambda_i + b_0 + 15 \\
(\sigma + 12)\lambda_i + b_0 - 5 \\
(\sigma^2 - b_0 - 11)\lambda_i + \sigma b_0 - 20
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-20 & 4 \\
b_0 + 15 & \sigma \\
b_0 - 5 & \sigma + 12 \\
\sigma b_0 - 20 & \sigma^2 - b_0 - 11
\end{bmatrix}
\begin{bmatrix}
1 \\
\lambda_i
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \begin{bmatrix}
\tilde{X}_1 \\
\tilde{X}_2
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
\lambda_1 & \lambda_2
\end{bmatrix}
= \begin{bmatrix}
\tilde{X}_1 \\
\tilde{X}_2
\end{bmatrix}
\]

— Separated
Step by Step (4)

\[ X = X_2X_1^{-1} = \tilde{X}_2 \mathcal{V} \mathcal{V}^{-1} \tilde{X}_1^{-1} = \tilde{X}_2 \tilde{X}_1^{-1} \]

\[ X = -\frac{1}{20\sigma + 4b_0 + 60} \times \]

\[ \begin{bmatrix}
20\sigma + 12b_0 + 180 & 20\sigma + 4b_0 + 220 \\
15\sigma^2 - b_0^2 + 20\sigma - 26b_0 - 165 & 20\sigma^2 + 4\sigma b_0 - 20b_0 - 300
\end{bmatrix} \]

\[ = -\frac{1}{\sigma^2 + 10\sigma + 2} \begin{bmatrix}
3\sigma^2 + 10\sigma + 6 & \sigma^2 + 10\sigma + 82 \\
\sigma^2 + 10\sigma + 82 & \sigma^3 + 5\sigma^2 - 28\sigma - 10
\end{bmatrix} \]
What Is the Advantage?

What We Get....

- Algebraic relationship between parameters and the SoR $\sigma$:
  \[
  \sigma^8 - 4(q_1^2 + 2)\sigma^6 + 2(3q_1^4 + 4q_1^2 + 8)\sigma^4 \\
  - 4(q_1^6 - 2q_1^4 + 256q_1^2q_2^2 + 8q_1^2)\sigma^2 + q_1^4(q_1 - 2)^2(q_1 + 2)^2 = 0
  \]

- Expression (rational function) for the Optimal Cost in terms of parameters and the SoR:
  E.g., $\mathcal{H}_2$ Optimal Cost — $\text{tr} \{ B^T XB \}$

What We Can Do....

Since we get an exact expression, we can....

- Take the derivative of $\sigma$ with respect to parameters: $\frac{\partial \sigma}{\partial q_i}$

- Take the derivative of the Optimal Cost with respect to parameters
Optimization of both the *Plant* and the *Controller*:

\[
\inf_{P \in \mathcal{P}, K \in \mathcal{K}} J(P, K) \rightarrow \inf_{P \in \mathcal{P}} \inf_{K \in \mathcal{K}} J(P, K)
\]

The ‘\(\inf_{K \in \mathcal{K}}\)’ part — *SoR Approach*

Newton's method applicable for the ‘\(\inf_{P \in \mathcal{P}}\)’ part
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Concluding Remarks

In This Talk....
- Solution approach to AREs by means of the SoR proposed
  - Polynomial spectral factorization by the SoR

In the Future....
- Solid and efficient implementation
  - Further structural properties exploited
- $\mathcal{H}_\infty$ control
- Discrete-time version
Thank you very much for your attention!