Solution of Algebraic Riccati Equations Using the Sum of Roots

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International Symposium on Symbolic and Algebraic Computation 2009

Korea Institute for Advanced Study (KIAS), Seoul, Korea

30 JULY 2009

What We Want to Do....

Gröbner Basis — *Algebraic method*

Computational Algebraic Geometry

Sum of Roots

Polynomial Spectral Factorization

Important Mathematical Tool for
Signal Processing and Control

Extend the result to solve *Algebraic Riccati Equations!*

Outline of the Talk

1. Polynomial Spectral Factorization via the Sum of Roots
2. Algebraic Riccati Equations
3. Sum of Roots Approach to Algebraic Riccati Equations
4. Concluding Remarks

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Polynomial Spectral Factorization

Problem Formulation

- **Given** : an even polynomial in λ (polynomial in λ^2)

$$f(\lambda) = \lambda^{2n} + a_{2n-2}\lambda^{2n-2} + a_{2n-4}\lambda^{2n-4} + \cdots + a_0$$

- **Assumption** : $f(\lambda)$ has no roots on the imaginary axis
- Roots of $f(\lambda)$: symmetrical w.r.t. the real & imaginary axes and the origin
- **Task** : Find a polynomial $g(\lambda)$ such that $\boxed{f(\lambda) = (-1)^n g(\lambda) g(-\lambda)}$ and $g(\lambda)$ has roots in the open *left half plane* only.
- $g(\lambda)$: monic without loss of generality

$$g(\lambda) = \lambda^n + \underbrace{\boldsymbol{\sigma}}_{-\sum_{i=1}^{n}\alpha_i}\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \cdots + b_0$$

- $\boldsymbol{\sigma}$: Sum of Roots — α_i : Roots of $f(\lambda)$ in the open *left half plane*

Solution Approach Using the Sum of Roots

$$g(\lambda) = \lambda^n + \boldsymbol{\sigma}\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \cdots + b_0$$

- $\boxed{f(\lambda) = (-1)^n g(\lambda) g(-\lambda)}$

  $\Longrightarrow$ a *system of algebraic equations* in $\boldsymbol{\sigma}$, b_j
- *Reduced Gröbner basis*
- *Efficient basis-conversion technique* $\Longrightarrow$ Shape basis obtained

Shape Basis

$$\begin{cases} S_f(\boldsymbol{\sigma}) = 0 & \text{— characteristic polynomial of } \sigma,\ 2^n\text{-th order} \\ b_{n-2} = h_{n-2}(\boldsymbol{\sigma}) & \text{— true } \boldsymbol{\sigma} = \text{largest real root of } S_f \\ b_{n-3} = h_{n-3}(\boldsymbol{\sigma}) & \\ \vdots & \\ b_0 = h_0(\boldsymbol{\sigma}) & \end{cases}$$

h_i: polynomial of order strictly less than 2^n

Outline of the Talk

Algebraic Riccati Equations

Problem Formulation

A quadratic matrix equation for an unknown $n \times n$ matrix X of the form

$$A^{\mathrm{T}}X + XA + XRX + Q = 0$$

- A, Q, R : given $n \times n$ matrices
- Q, R : symmetric

- Positive semi-definite X ($X = X^{\mathrm{T}} \geq 0$) to be found

Assumptions

- Q positive semi-definite, R negative semi-definite
- (A, R) stabilizable, (Q, A) detectable

$\Longrightarrow$ *Guarantees no roots on the imaginary axis and the existence of positive semi-definite X*

Target

Naïve Approach

- Elements of $X = (x_{i,j})$: variables
- $A^{\mathrm{T}}X + XA + XRX + Q = 0$
  $\Longrightarrow$ A set of algebraic equations in $x_{i,j}$
- Employ general Gröbner basis computation algorithms

Drawbacks

- Cannot make use of structural properties
- # of variables : $\frac{1}{2}n(n+1)$
- Difficult to characterize the positive semi-definite solution (?)

Aim

- Characterize the solution X by means of the SoR

Standard (Numerical) Solution Approach

$$A^{\mathrm{T}}X + XA + XRX + Q = 0$$

1. Construct the *Hamiltonian matrix*

$$H := \begin{bmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{bmatrix} \qquad (2n \times 2n \text{ matrix})$$

2. Compute n eigenvectors v_i
   associates with the eigenvalues of H in the left half plane.
3. $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \qquad X_1, X_2 : n \times n$ matrices
4. $X = X_2 X_1^{-1} (\geq 0)$

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Proposed Approach (1)

Input

- $A, Q, R \in \mathbb{Q}[\mathbf{q}]^{n\times n}$
- $\mathbf{q} = (q_1, \ldots, q_m)$: vector of parameters

Output

- X : rational functions of parameters $\mathbf{q}$ and the SoR σ
- $S_f(\sigma)$: Characteristic polynomial relating parameters $\mathbf{q}$ and the SoR σ

! Closed-form expression for X in terms of *parameters* — *not sought / impossible*

Proposed Approach (2)

Consists of 4 Steps....

1. Compute the characteristic polynomial $f(\lambda)$ of H. Carry out polynomial spectral factorization for $f(\lambda) = (-1)^n g(\lambda) g(-\lambda)$.
   — *Relate parameters and the SoR*
2. Express the eigenvector $v(\lambda')$ as polynomials in the eigenvalue λ'
3. Express $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ as a product of two matrices: $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} V$
   $\tilde{X}_1, \tilde{X}_2$: matrices in the coefficients of $g(\lambda)$ — *SoR*
   V : matrices in eigenvalues λ_i
   — *Separate σ, b_j and λ_i*
4. $X = \tilde{X}_2 \tilde{X}_1^{-1}$

Step by Step (1)

$$A = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

1. $$H = \begin{bmatrix} A & R \\ -Q & -A^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} -2 & -3 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -10 & -14 & 2 & -1 \\ -14 & -20 & 3 & 0 \end{bmatrix}$$

   Characteristic polynomial $f(\lambda) = \det(\lambda I_4 - H) = \lambda^4 - 28\lambda^2 + 35$

   Polynomial spectral factorization for $f(\lambda)$:

   $$g(\lambda) = \lambda^2 + \sigma\lambda + b_0$$

   $$\boxed{f(\lambda) = (-1)^n g(\lambda) g(-\lambda)} \quad \Longrightarrow \quad \begin{cases} \sigma^4 - 56\sigma^2 + 644 = 0 \\ b_0 = \frac{1}{2}\sigma^2 - 14 \end{cases}$$

Step by Step (2)

2. Get an expression for the eigenvector.
   Compute the quotient: $f(\lambda) = (\lambda - \lambda')q(\lambda) + r$

$$f(H) = (H - \lambda' I_{2n})q(H) = 0 \implies H \underbrace{q(H)u}_{v(\lambda')} = \lambda' \underbrace{q(H)u}_{v(\lambda')}$$

for suitable chosen u

$$q(\lambda) = \lambda^3 + \lambda'\lambda^2 + (\lambda'^2 - 28)\lambda + \lambda'^3 - 28\lambda'$$

$$u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \implies v(\lambda') = q(H)u = \begin{bmatrix} 4\lambda' - 20 \\ -\lambda'^2 + 15 \\ -\lambda'^2 + 12\lambda' - 5 \\ \lambda'^3 - 11\lambda' - 20 \end{bmatrix}$$

! *The quotient to be computed only once!*

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Step by Step (3)

3. $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} v(\lambda_1) & v(\lambda_2) & \cdots & v(\lambda_n) \end{bmatrix}$$

— λ_i : eigenvalues in the left half plane

In fact, it suffices to consider $\boxed{v(\lambda_i) \mod g(\lambda_i)}$

$$v(\lambda_i) \mod g(\lambda_i) = \left[\begin{array}{c} 4\lambda_i - 20 \\ \sigma\lambda_i + b_0 + 15 \\ \hline (\sigma + 12)\lambda_i + b_0 - 5 \\ (\sigma^2 - b_0 - 11)\lambda_i + \sigma b_0 - 20 \end{array}\right]$$

$$= \left[\begin{array}{cc} -20 & 4 \\ b_0 + 15 & \sigma \\ \hline b_0 - 5 & \sigma + 12 \\ \sigma b_0 - 20 & \sigma^2 - b_0 - 11 \end{array}\right] \begin{bmatrix} 1 \\ \lambda_i \end{bmatrix} = \left[\begin{array}{c} \tilde{X}_1 \\ \hline \tilde{X}_2 \end{array}\right] \begin{bmatrix} 1 \\ \lambda_i \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} V$$ — *Separated*

Step by Step (4)

4. $X = X_2 X_1^{-1} = \tilde{X}_2 V V^{-1} \tilde{X}_1^{-1} = \tilde{X}_2 \tilde{X}_1^{-1}$ — rational functions in σ, b_i

$$X = -\frac{1}{20\sigma + 4b_0 + 60} \times \begin{bmatrix} 20\sigma + 12b_0 + 180 & 20\sigma + 4b_0 + 220 \\ 15\sigma^2 - b_0{}^2 + 20\sigma - 26b_0 - 165 & 20\sigma^2 + 4\sigma b_0 - 20b_0 - 300 \end{bmatrix}$$

$$= -\frac{1}{\sigma^2 + 10\sigma + 2} \begin{bmatrix} 3\sigma^2 + 10\sigma + 6 & \sigma^2 + 10\sigma + 82 \\ \sigma^2 + 10\sigma + 82 & \sigma^3 + 5\sigma^2 - 28\sigma - 10 \end{bmatrix}$$

What Is the Advantage?

What We Get....

- Algebraic relationship between *parameters* and the *SoR* σ:

$$\sigma^8 - 4(q_1^2 + 2)\sigma^6 + 2(3q_1^4 + 4q_1^2 + 8)\sigma^4 \\ - 4(q_1^6 - 2q_1^4 + 256q_1^2q_2^2 + 8q_1^2)\sigma^2 + q_1^4(q_1 - 2)^2(q_1 + 2)^2 = 0$$

- Expression (rational function) for the *Optimal Cost* in terms of *parameters* and the *SoR*

  E.g., $\mathcal{H}_2$ Optimal Cost — $\mathrm{tr}\left\{B^{\mathrm{T}}XB\right\}$

What We Can Do....

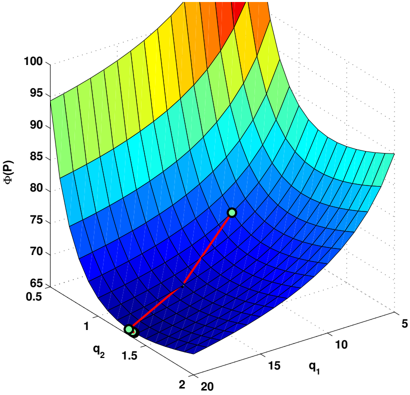
Since we get an *exact* expression, we can....

- Take the derivative of σ with respect to parameters: $\dfrac{\partial \sigma}{\partial q_i}$
- Take the derivative of the *Optimal Cost* with respect to parameters

- Optimization of both the *Plant* and the *Controller*:

$$\inf_{P\in\mathcal{P},\, K\in\mathcal{K}} J(P,\, K) \quad \Longrightarrow \quad \inf_{P\in\mathcal{P}} \inf_{K\in\mathcal{K}} J(P,\, K)$$

- The '$\inf_{K\in\mathcal{K}}$' part — *SoR Approach*
- Newton's method applicable for the '$\inf_{P\in\mathcal{P}}$' part



Outline of the Talk

Concluding Remarks

In This Talk....

- Solution approach to AREs by means of the SoR proposed
  - Polynomial spectral factorization by the SoR

In the Future....

- Solid and efficient implementation
  - Further structural properties exploited
- $\mathcal{H}_\infty$ control
- Discrete-time version

Thank you very much for your attention!