A Non-Holonomic Systems Approach to Special Function Identities

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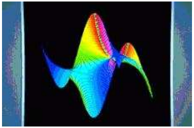

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INRIA
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(supported in part by the Microsoft Research – INRIA Joint Centre)

July 29, 2009

I Introduction

Goal: Identities in Special Functions

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{\infty} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$ [Strehl 1992] binomial nbs

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\, dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$ [GlMo 1994] Bessel fns

$$\frac{1}{2\pi i} \oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\, dy = \frac{H_n(x)}{\lfloor n/2 \rfloor !}$$ [Doetsch 1930] orthog. poly.

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n$$ [Abel 1826] k^k

$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle$$ [Frobenius 1910] Stirling 2nd kind, Eulerian nbs

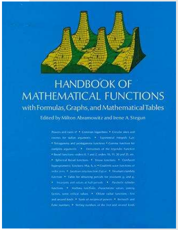
$$\sum_{k=0}^{\infty} \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^{\infty} \binom{n}{k} B_{m+k}$$ [Gessel 2003] Bernoulli nbs

$$\int_0^{\infty} x^{k-1} \zeta(n, \alpha+\beta x)\, dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha)$$ Hurwitz's fn

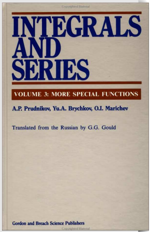
$$\int_0^{\infty} x^{\alpha-1} \operatorname{Li}_n(-xy)\, dx = \frac{\pi(-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$ polylogarithm

$$\int_0^{\infty} x^{s-1} \exp(xy) \Gamma(a, xy)\, dx = \frac{\pi y^{-s}}{\sin\big((a+s)\pi\big)} \frac{\Gamma(s)}{\Gamma(1-a)}$$ incomplete gamma fn

Equations are Better than Closed Forms!



linear differential/recurrence equations
+ initial conditions = data structure



Parametrized summation and integration algorithms

- (q-)hypergeometric/hyperexponential: Zeilberger (1990), Paule–Schorn (1995); Riese (2003); Almkvist–Z. (1990)
- higher-order eqns: Zeilberger (1990), Takayama (1989–90), FC–BS (1998), FC (2000)
- Abel-type/Stirling/Euler and Bernoulli: Majewicz (1996); Kauers (2007); Chen & Sun (2009)
- previous classes and more: the present work

Three Ideas

Confinement (in finite-dim'l v.-s., resp. positive-dim'l modules)

"Higher-order" derivatives, $\partial^s(f)$, rewrite as linear combinations of a specific set of "lower-order" derivatives, $\partial^a(f), a \in A$.

More than $\#A$ derivatives $\to$ equation(s). OLD/NEW

Polynomial growth (degree bound on coeffs of normal forms)

$$i \leq s \quad \Rightarrow \quad \partial^i(f) = \frac{1}{P_s} \sum_{a \in A} (\text{degree in } y \leq O(s^p)) \partial^a(f)$$

Sufficiently small $p \to$ equations(s) free of y. NEW

Creative telescoping ($\approx$ diff. under the int. sign + int. by parts)

Equation free of $y \to$ equation on integral/sum w.r.t. y.

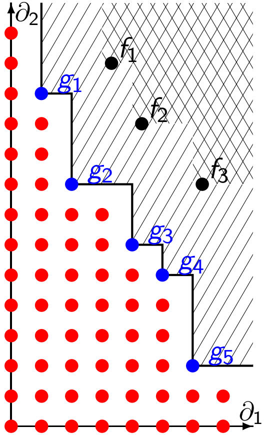
Skew polynomial elimination or variant approaches. OLD

II Algebraic Closures

Dimension of Ideals and ∂-Finite Functions

Several eqns $\longrightarrow$ left Gröbner bases in $\mathcal{R} = \mathbb{Q}(x,\dots)\langle\partial,\dots\rangle$.

$M_s(I) := \{\, m : m$ is below the stairs and of total degree $\leq s \,\}$.



$\mathrm{GB}(f_1,\dots) = (g_1,\dots)$

Theorem (Hilbert) & Definition

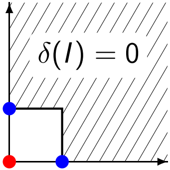
- For any I, there is an integer $\delta(I)$ such that $\#M_s(I) = O\big(s^{\delta(I)}\big)$.
- $\delta(I)$ is the (Hilbert) dimension of I.

Definition (annihilator and ∂-finiteness)

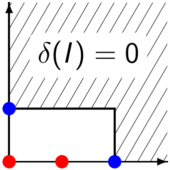
- $\operatorname{ann} f := \{\, P \in \mathcal{R} : P(f) = 0 \,\}$
- f is ∂-finite $\Leftrightarrow$ $\delta(\operatorname{ann} f) = 0$

Examples

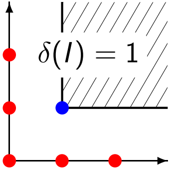
Binomial coeffs $\binom{n}{k}$ w.r.t. S_n, S_k;
Hypergeometric sequences:



Bessel $J_\nu(x)$ w.r.t. S_ν, D_x;
Orthogonal polys w.r.t. S_n, D_x:

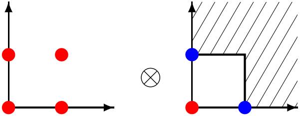


Stirling nbs w.r.t. S_n, S_k:

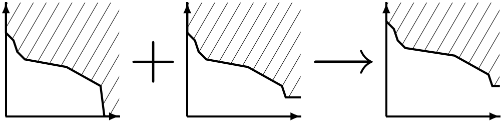


Abel-type w.r.t. S_m, S_k, S_r, S_s:

$\delta(I) = 2$



Algebraic Closure Properties in Positive Dimension



Proposition NEW

δ of sum $\leq$ max. of δ's, δ of product $\leq$ sum of δ's, δ of der. $\leq \delta$.

Algorithm (for a product fg): for a *graded* ordering, NEW

for $s = 0, 1, 2, \dots,$ until $\delta(I) \leq$ bound:
for each $|\alpha| \leq s$, reduce $\partial^\alpha(fg)$ to a sum $\sum u_{\alpha;\beta,\gamma}(x)\partial^\beta(f)\partial^\gamma(g)$
over $\beta \in M_s(\operatorname{ann} f), \gamma \in M_s(\operatorname{ann} g)$
search for $Q(x)$-linear relations, set I to the ideal they generate
return I, a subideal of $\operatorname{ann} fg$

Example (Stirling numbers of the second kind): $\left\{{n \atop k}\right\}$, $\delta(I) = 1$,
1st-order rec. $\xrightarrow{s=3}$ $\left\{{n \atop k}\right\}\left\{{m \atop k}\right\}$, $\delta(I) = 2$, 2nd-order rec.

III Closures under $\sum$ and $\int$

Creative Telescoping (Zeilberger, 1990)

$$U(x) = \int_a^b u(x,y)\,dy = \,?$$

Given $A(x, D_x)$ and $B(x, y, D_x, D_y)$ such that

$$\big(A(x, D_x) - D_y B(x, y, D_x, D_y)\big)(u) = 0,$$

integration leads by “telescoping” to

$$A(x, D_x)(U) = \big[B(x, y, D_x, D_y)(u)\big]_{y=a}^{y=b} \overset{\text{often}}{=} 0.$$

Definition (telescoping ideal of I w.r.t. y)

$$T_y(I) := \big(I + \partial_y \mathcal{R}_{x,y}\big) \cap \mathcal{R}_x \quad \text{where}$$

$$\mathcal{R}_{x,y} := K(x,y)\langle \partial_x, \partial_y \rangle \text{ and } \mathcal{R}_x := K(x)\langle \partial_x \rangle.$$

$$U_n = \sum_{k=a}^{b} u_{n,k}, \quad U(x) = \sum_{k=a}^{b} u_k(x), \quad U_n = \int_a^b u_n(y)\,dy.$$

Polynomial Growth and Creative Telescoping

Definition: an ideal has polynomial growth p w.r.t. y ... NEW

...if there exist polynomials $P_s(x, y)$, s.t. if $|a| + b \leq s$,

$$P_s \partial_{x_1}^{a_1} \ldots \partial_{x_\ell}^{a_k} \partial_y^b \text{ reduces to polys of degree } O(s^p) \text{ in } y.$$

- $f = \binom{n}{k}$: $P_s = (k+1)_s (n+1-k)_s$ confines shifts of order $\leq s$ in dim. $O(s^1)$ over $K(n)$ → Pascal's formula.
- $f = \frac{a(x,y_1,\ldots,y_r)}{b(x,y_1,\ldots,y_r)}$: $P_s = b^{s+1}$ confines derivatives of order $\leq s$ in dim. $O(s^r)$ over $K(x)$ → elimination of $y_1, \ldots, y_r$ possible.

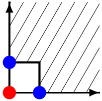
Theorem: $\delta(T_y(I)) \leq \max\big(\delta(I) + p - 1, 0\big)$. NEW

Corollary (sufficient condition for creative telescoping) NEW

$$\delta(I) + p - 1 < \ell \Rightarrow \text{identities exist for the sum/int. w.r.t. } y.$$

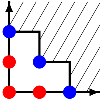
Examples with Polynomial Growth $p = 1$

- Proper hypergeometric (Wilf & Zeilberger, 1992):

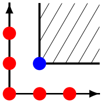


$$Q(n,k)\xi^k \frac{\prod_{i=1}^{u}(a_i n + b_i k + c_i)!}{\prod_{i=1}^{v}(u_i n + v_i k + w_i)!},$$

Q polynomial, a_i, b_i, u_i, v_i integers.



- Differentially finite ("holonomic"; Takayama, 1992).
- Stirling: $\delta = 1 \to$ for ≥ 3 vars, e.g., Frobenius:



$$\sum_{k=0}^{n} (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ {n+1 \atop k+1} \right\} = \left\langle {n \atop m} \right\rangle.$$

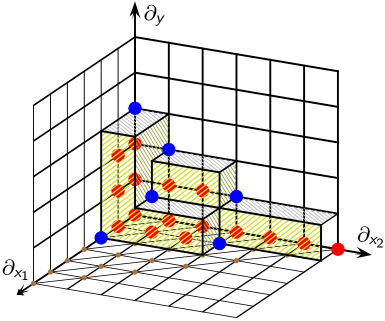
- Abel type: $\delta = 2 \to$ for ≥ 4 vars, e.g., Abel:

$$\sum_{k=0}^{n} \binom{n}{k} i(k+i)^{k-1}(n-k+j)^{n-k} = (n+i+j)^n.$$

Zeilberger's Fast Algorithm Extended to $\delta > 0$

NEW

1. (q-)Hypergeometric: Zeilberger 1990 (impl.: Schorn, Riese).
2. ∂-finite ($\delta = 0$): Chyzak 2000 (impl.: Koutschan, Pech).
3. Non-∂-finite (NEW): for a *graded* ordering,



for $s = 0, 1, 2, \dots,$ until $\delta(J) \leq$ bound:
  set $A := \sum_{|\alpha| \leq s} \eta_\alpha(x) \partial^\alpha$
    for undetermined coeffs η_α
  set $B := \sum_{\beta \in M_{s-1}(I)} \phi_\beta(x, y) \partial^\beta$
    for undetermined coeffs $\phi_\beta(y)$
  reduce $A - \partial_y B$ onto the basis $M_s(I)$
  extract coeffs to form a linear system
    of first order w.r.t. ∂_y
  solve and set J to the ideal of the A's
return the pairs (A, B)

IV Conclusion

Conclusion

- Summary:
  - Linear differential/recurrence equations as a data structure;
  - Confinement + polynomial growth + creative telescoping $\rightarrow$ identities;
  - Input dimension + polynomial growth $\rightarrow$ output dimension.
- Also in this work:
  - Fasenmyer's style algorithm possible;
  - Multiple summation/integration.
- Future & Open questions:
  - Hilbert-driven approach should be possible;
  - Replace polynomial growth by something intrinsic;
  - Bounds $\rightarrow$ identities + their size + complexity of algorithms;
  - Exploit symmetries;
  - Structured Padé-Hermite approximants;
  - Understand non-minimality.