Liouvillian Solutions of Irreducible Linear Difference Equations

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Talk presented by YONGJAE CHA

Liouvillian Solutions of Linear Difference Equations: Algorithms

1. P. A. Hendriks and M. F. Singer, 1999
   - Definition of Liouvillian solutions, and the first algorithm to compute them.
2. R. Bomboy, 2002
3. D.E. Khmelnov, 2008
4. R. Feng, M. F. Singer, M. Wu, 2008
5. S.A. Abramov, M.A. Barkatou and D.E. Khmelnov, 2009
6. Y. Cha and M. van Hoeij, 2009
   - Reduced combinatorial complexity (but only the irreducible case is handled).

Liouvillian Solutions of Linear Difference Equations: Our Contributions

- Prior algorithms reduce computing:
  Liouvillian solutions of L
  to a previously solved problem:
  Hypergeometric solutions of another operator, say $\tilde{L}$.

- Hypergeometric solutions are computed with a combinatorial algorithm (cost is exponential in # singularities).

- Problem: $\tilde{L}$ has n times more singularities than L
  (this raises # combinations to the n'th power!)
- Our algorithm does not increase the number of singularities.
  (so # combinations is smaller).

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Liouvillian Solutions of Linear Difference Equations: Linear Difference Operator

A linear difference operator

$$L = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_0\tau^0$$

where $a_i \in \mathbb{C}(x)$ and τ is the shift operator: $\tau(u(x)) = u(x+1)$ corresponds to a difference equation

$$a_n(x)u(x+n) + a_{n-1}(x)u(x+n-1) + \cdots + a_0(x)u(x) = 0.$$

Example:

- If $L = \tau - x$ then the equation $L(u(x)) = 0$ is $u(x+1) - xu(x) = 0$ and $\Gamma(x)$ is a solution of L.

Gauge Equivalence

Notation:

- $V(L)$ = solution space of L.

Definition

Operators L_1 and L_2 in $\mathbb{C}(x)[\tau]$ are called *gauge equivalent* if they have the same order and

$$G(V(L_1)) = V(L_2) \text{ for some } G \in \mathbb{C}(x)[\tau].$$

Then G is called a *gauge transformation* from L_1 to L_2.

Inverse gauge transformation:

- Given L_1 and G we can find $G' \in \mathbb{C}(x)[\tau]$ such that $G'(V(L_2)) = V(L_1)$.

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Gauge Equivalence

Notation:

- $L_1 \sim_g L_2$ means L_1 is gauge equivalent to L_2.

Remark

If $L_1 \sim_g L_2$ and if we can solve L_1 then we can also solve L_2.

1. Find gauge transformation G with existing software,
2. then apply G to solutions of L_1 to get solutions of L_2.

Liouvillian Solutions of Linear Difference Equations: Property

Theorem (Hendriks Singer 1999)

If $L = a_n\tau^n + \cdots + a_0\tau^0$ *is irreducible then*

$\exists$ *Liouvillian Solutions* $\Longleftrightarrow \exists b_0 \in \mathbb{C}(x)$ *such that*

$$a_n\tau^n + \cdots + a_0\tau^0 \quad \sim_g \quad \tau^n + b_0\tau^0$$

Remark

Operators of the form $\tau^n + b_0\tau^0$ are easy to solve, so if we know b_0 then we can solve L.

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Liouvillian Solutions of Linear Difference Equations: The Problem

Let $L = a_n\tau^n + \cdots + a_0\tau^0$ with $a_i \in \mathbb{C}[x]$ and assume that

$$L \sim_g \tau^n + b_0\tau^0$$

for some unknown $b_0 \in \mathbb{C}(x)$.

If we can find b_0 then we can solve $\tau^n + b_0\tau^0$ and hence solve L.

Notation

write $b_0 = c\phi$ where $\phi = \frac{\text{monic poly}}{\text{monic poly}}$ and $c \in \mathbb{C}^*$.

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c is easy to compute, the main task is to compute ϕ.

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Liouvillian Solutions of Linear Difference Equations: Approach

Definition

Let $L = a_n\tau^n + \cdots + a_0\tau^0 \in \mathbb{C}[x][\tau]$ then the finite singularities of L are $Sing = \{q + \mathbb{Z} \in \mathbb{C}/\mathbb{Z} \mid \text{q is root of } a_0 a_n\}$

Theorem

If $q_1 + \mathbb{Z}, \ldots, q_k + \mathbb{Z}$ are the finite singularities then we may assume $\phi = \prod_{i=1}^{k}\prod_{j=0}^{n-1}(x - q_i - j)^{k_{i,j}}$ *with* $k_{i,j} \in \mathbb{Z}$.

1. At each finite singularity $p_i \in \mathbb{C}/\mathbb{Z}$ (where $p_i = q_i + \mathbb{Z}$) we have to find n unknown exponents $k_{i,0}, \ldots, k_{i,n-1}$.
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Valuation Growth

Definition

Let $u(x) \in \mathbb{C}(x)$ be a non-zero meromorphic function. The *valuation growth* of $u(x)$ at $p = q + \mathbb{Z}$ is

$$\liminf_{n\to\infty}(\text{order of } u(x) \text{ at } x = n + q) - \liminf_{n\to\infty}(\text{order of } u(x) \text{ at } x = -n + q)$$

Definition

Let $p \in \mathbb{C}/\mathbb{Z}$ and L be a difference operator. Then $\mathrm{Min}_p(L)$ resp. $\mathrm{Max}_p(L)$ is the minimum resp. maximum valuation growth at p, taken over all meromorphic solutions of L.

Theorem

If $L_1 \sim_g L_2$ then they have the same $\mathrm{Min}_p, \mathrm{Max}_p$ for all $p \in \mathbb{C}/\mathbb{Z}$.

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Example of Operator of order 3 with one finite singularity at $p = \mathbb{Z}$

Suppose $L = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$ and that

$$L \sim_g \tau^3 + c \cdot x^{k_0}(x-1)^{k_1}(x-2)^{k_2}$$

1. c can be computed from a_0/a_3
2. $k_0 + k_1 + k_2$ can be computed from a_0/a_3
3. $\max\{k_0, k_1, k_2\} = \mathrm{Max}_{\mathbb{Z}}(L)$
4. $\min\{k_0, k_1, k_2\} = \mathrm{Min}_{\mathbb{Z}}(L)$

Items 2, 3, 4 determine k_0, k_1, k_2 up to a permutation.

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Example with two finite singularities at $\mathbb{Z}$ and $\frac{1}{2}+\mathbb{Z}$

Suppose $L = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$ is gauge equivalent to

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This determines k_0,k_1,k_2 up to a permutation, and also l_0,l_1,l_2 up to a permutation.

Worst case is $3!\cdot 3!$ combinations (actually: 1/3 of that).

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Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

- $Sing = \{\mathbb{Z}, \frac{1}{2} + \mathbb{Z}\}$ and $c = -2$.
- At $\mathbb{Z}$,

$$min = 0, \quad max = 1, \quad sum = 2$$

  So the exponents of $x^{\cdots}(x-1)^{\cdots}(x-2)^{\cdots}$ must be a permutation of $0, 1, 1$
- At $\frac{1}{2} + \mathbb{Z}$,

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Candidates of $c\phi$ are

1. $-2x^1(x-1)^1(x-2)^0(x-1/2)^1(x-3/2)^0(x-5/2)^0$
2. $-2x^1(x-1)^1(x-2)^0(x-1/2)^0(x-3/2)^1(x-5/2)^0$
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9. $-2x^1(x-1)^0(x-2)^1(x-1/2)^0(x-3/2)^1(x-5/2)^0$

Only need to try 1, 2, 3, the others are redundant.

Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

- $\tau^3 - 2x(x-1)(x-1/2)$ is gauge equivalent to L
- Gauge transformation is $\tau + x - 1$.
- Basis of solutions of $\tau^3 - 2x(x-1)(x-1/2)$ is

$$\{(\xi^k)^x v(x)\} \quad \text{for} \quad k = 0 \dots 2$$

where $v(x) = 3^x 2^{x/3} \Gamma(\frac{x}{3}) \Gamma(\frac{x-1}{3}) \Gamma(\frac{x-\frac{1}{2}}{3})$ and $\xi^3 = 1$.

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- Basis of solutions of $\tau^3 - 2x(x-1)(x-1/2)$ is

$$\{(\xi^k)^x v(x)\} \quad \text{for} \quad k = 0 \ldots 2$$

where $v(x) = 3^x 2^{x/3} \Gamma(\frac{x}{3}) \Gamma(\frac{x-1}{3}) \Gamma(\frac{x-\frac{1}{2}}{3})$ and $\xi^3 = 1$.

- Thus, Basis of solutions of L is

$$\{(\xi^k)^{x+1} v(x+1) + (x-1)(\xi^k)^x v(x)\} \quad \text{for} \quad k = 0 \ldots 2$$