Fast Simplications for Tarski Formulas

C. W. Brown

Department of Computer Science
U. S. Naval Academy

International Symposium on Symbolic and Algebraic Computation, 2009
Computing with real polynomial inequalities/equalities

\[ S := (2x + y - 1)(2x - y - 1) < 0 \land x^2 + y^2 - 1 \leq 0 \lor x - 1 = 0 \]
Computing with real polynomial inequalities/equalities

\[ S := (2x + y - 1)(2x - y - 1) < 0 \land x^2 + y^2 - 1 \leq 0 \lor x - 1 = 0 \]

Some kinds of questions we might want to answer

<table>
<thead>
<tr>
<th>SAT</th>
<th>QE</th>
<th>Optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is S satisfiable?</td>
<td>Describe ( x ) s.t. ( \exists y[S] )</td>
<td>maximize ( \frac{xy-1}{x^2+y^2} ) over ( S )</td>
</tr>
<tr>
<td>Yes, for example at ((-13/16, 1/4))</td>
<td>( x + 1 \geq 0 \land x + 4/9 \leq 0 \lor 17x^2 + 16x + 3 &lt; 0 \lor x - 1 = 0 )</td>
<td>( (\sqrt{2} - 1)/2 )</td>
</tr>
</tbody>
</table>
Computing with real polynomial inequalities/equalities

\[ S := (2x + y - 1)(2x - y - 1) < 0 \land x^2 + y^2 - 1 \leq 0 \lor x - 1 = 0 \]

Some kinds of questions we might want to answer

<table>
<thead>
<tr>
<th>SAT</th>
<th>QE</th>
<th>Optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is ( S ) satisfiable?</td>
<td>Describe ( x ) s.t. ( \exists y [S] )</td>
<td>maximize ( \frac{xy - 1}{x^2 + y^2} ) over ( S )</td>
</tr>
<tr>
<td>Yes, for example at ((-13/16, 1/4))</td>
<td>( x + 1 \geq 0 \land x + 4/9 \leq 0 \lor 17x^2 + 16x + 3 &lt; 0 \lor x - 1 = 0 )</td>
<td>( (\sqrt{2} - 1)/2 )</td>
</tr>
</tbody>
</table>

Problems with symbolic methods:

- slow
- memory intensive
- sensitive to input presentation
- overly complex results
Overview

Simplification of polynomial eq’s and ineq’s

Kinds: SAT is simplification; minimize w.r.t. to some metric; more explicit is simpler; “I know it when I see it”

Prior Work: Limited!

- Hong 1992
- Dolzman & Sturm 1997
- Brown 1999

My goal: Simplify quickly! We need algorithms for simplifications that can be done quickly, and proofs for simplifications that can’t.

This paper: Defines subclass of simplifications, provides efficient algorithms for some, and intractability proofs for others.
Using propositional logic to simplify inequalities

\[ x^2 - 3xyz > 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]
\[ \lor \]
\[ x^2 - 3xyz \leq 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]
Using propositional logic to simplify inequalities

\[ x^2 - 3xyz > 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]
\[ \lor \]
\[ x^2 - 3xyz \leq 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]

↓ map to propositional logic

\[ A \land B \land C \lor \neg A \land B \land C \]
Using propositional logic to simplify inequalities

\[ x^2 - 3xyz > 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]
\[ \lor \]
\[ x^2 - 3xyz \leq 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]

↓ map to propositional logic

\[ A \land B \land C \lor \neg A \land B \land C \]

↓ simplify

\[ B \land C \]
Using propositional logic to simplify inequalities

\[ x^2 - 3xyz > 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]
\[ \lor \]
\[ x^2 - 3xyz \leq 0 \land z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]

↓ map to propositional logic

\[ A \land B \land C \lor \neg A \land B \land C \]

↓ simplify

\[ B \land C \]

↓ map back

\[ z^3y + x < 0 \land y^2z^2 - 2x + 1 > 0 \]
Using propositional logic to simplify inequalities (cont.)

The good

- Satisfiability and simplification for propositional logic is well-studied; algorithms exist with good practical performance.
- The inequalities are complete black boxes — time to check SAT or simplify is independent of degree, bit-size, and other metrics for size of the individual inequalities. No algebraic computations!

The bad

- Deductions require repetitions of inequalities (or their negations).
- The inequalities are complete black boxes — cannot make deductions based on them.
- SAT is NP-Complete, Simplification is $\Sigma_2^P$-Complete.
Big idea: map to the theory of “monomial inequalities”

\[(x + y)^3 x < 0 \land (x + y)^2 x (z^2 - 2xy) < 0 \land (x + y)x^5(z^2 - 2xy) > 0\]
Big idea: map to the theory of “monomial inequalities”

\[(x + y)^3 x < 0 \land (x + y)^2 x(z^2 - 2xy) < 0 \land (x + y)x^5(z^2 - 2xy) > 0\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[x_1^3 x_2 < 0 \land x_1^2 x_2 x_3 < 0 \land x_1 x_2^5 x_3 > 0\]
Big idea: map to the theory of “monomial inequalities”

\[(x + y)^3 x < 0 \land (x + y)^2 x(z^2 - 2xy) < 0 \land (x + y)x^5(z^2 - 2xy) > 0\]

\[\downarrow \quad \downarrow \quad \downarrow\]

\[x_1^3 x_2 < 0 \land x_1^2 x_2 x_3 < 0 \land x_1 x_2^5 x_3 > 0\]

\[\downarrow\]

\[x_1 < 0 \land x_2 > 0 \land x_3 < 0\]
Big idea: map to the theory of “monomial inequalities”

\[(x + y)^3 x < 0 \land (x + y)^2 x(z^2 - 2xy) < 0 \land (x + y)x^5(z^2 - 2xy) > 0\]

\[\downarrow \quad \downarrow \quad \downarrow\]

\[x_1^3 x_2 < 0 \land x_1^2 x_2 x_3 < 0 \land x_1 x_2^5 x_3 > 0\]

\[\downarrow\]

\[x_1 < 0 \land x_2 > 0 \land x_3 < 0\]

\[\downarrow \quad \downarrow \quad \downarrow\]

\[x + y < 0 \land x > 0 \land z^2 - 2xy < 0\]
Restriction to conjunctions of monomial inequalities

- Allowing disjunctions means our SAT problem is NP-Hard, and simplification is $\Sigma^P_2$-Hard.
- Expressiveness of conjunctions of monomial inequalities lies strictly between that of conjunctions of single-variable sign conditions and of arbitrary boolean combinations of sign-conditions.
  - $x > 0 \land y > 0 \lor x = 0$ cannot be expressed as a conjunction of monomial inequalities
  - $xy < 0$ cannot be expressed as a conjunction of single-variable sign conditions.
- Conjunction of monomial-inequalities is often a compact representation.
- Simplifications still useful for formulas with disjunction.
Results for conjunctions of monomial inequalities

- Algorithm for SAT for conjunctions of monomial inequalities; runs in (low degree) polynomial time.
- Proof that simplification of conjunctions of monomial inequalities (minimizing sum of total-degrees) is NP-Hard (decision-problem variant is NP-Complete)
- Proof that simplifying a single monomial inequality \( p \sigma 0 \) assuming some conjunction of monomial inequalities \( F \) holds is NP-Hard.
- Algorithm for determining all single-variable sign-conditions implied by a given conjunction of monomial inequalities; runs in (low degree) polynomial time.
- Algorithm for determining all equalities implied by a given conjunction of monomial inequalities; runs in (low degree) polynomial time.
Restricting to strict monomial inequalities

Consider the following mapping from strict monomial inequalities in \( \{x_1, \ldots, x_n\} \) and \( GF(2)^{n+1} \):

\[
\Gamma \left( \prod_{j=1}^{n} x_j^{d_j} \sigma 0 \right) = [B_1, \ldots, B_{n+1}],
\]

where \( B_j = d_j \mod 2 \), for \( 1 \leq j \leq n \), and \( B_{n+1} = 0 \) if \( \sigma \) is > and 1 if \( \sigma \) is <.

Examples:

- \( \Gamma(x_1 x_3 < 0) = [1, 0, 1, 1] \)  
- \( \Gamma(x_1 x_3 > 0) = [1, 0, 1, 0] \)  
- \( \Gamma(x_1^2 x_2^3 x_3 < 0) = [0, 1, 1, 1] \)  

\( \Gamma^{-1}([1, 0, 1, 1]) = x_1 x_3 < 0 \)  
\( \Gamma^{-1}([0, 0, 0, 0]) = 1 > 0 \)  
\( \Gamma^{-1}([0, 0, 0, 1]) = 1 < 0 \)
**Main theorem**

**Theorem:** Let $F = A_1 \land \cdots \land A_m$, where each $A_i$ is a monomial inequality over $\{x_1, \ldots, x_n\}$, and let $b_i = \Gamma(A_i)$. Assuming that $F$ is satisfiable over $\mathbb{R}$,

1. $[0, \ldots, 0, 1] \notin \text{span}(b_1, \ldots, b_m)$, and
2. $b \in \text{span}(b_1, \ldots, b_m)$ if and only if $\forall x_1, \ldots, x_n[F \Rightarrow \Gamma^{-1}(b)]$.

**Corollary:** $F$ is satisfiable over $\mathbb{R}$ iff $[0, \ldots, 0, 1] \notin \text{span}(b_1, \ldots, b_m)$. 

---

**Fast Simplications for Tarski Formulas**

ISSAC 2009 10 / 13
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable?
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable?

Routh-Hurwitz criterion gives

\[
\frac{73}{62} x^2 y > 0 \land 3844x^3(3472x^2 + 7081y^3)y > 0 \land \\
\frac{1}{62} (3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) > 0 \land \\
-1/87 (-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) > 0
\]
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable?

Routh-Hurwitz criterion gives

\[
\begin{align*}
73/62 \ x^2y & > 0 \land 3844x^3(3472x^2 + 7081y^3)y > 0 \land \\
1/62 (3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) & > 0 \\
\land -1/87 (-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) & > 0
\end{align*}
\]

Combinatorial part: \( x_1^2x_2 > 0 \land x_1^3x_2x_3 > 0 \land x_3^2x_2x_1^2x_4 > 0 \land x_2x_3x_4 < 0 \)
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable? Routh-Hurwitz criterion gives

\[
\frac{73}{62} x^2 y > 0 \land 3844x^3(3472x^2 + 7081y^3)y > 0 \land \\
\frac{1}{62} (3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) > 0 \\
\land - \frac{1}{87}(-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) > 0
\]

Combinatorial part: \( x_1^2x_2 > 0 \land x_1^3x_2x_3 > 0 \land x_3^2x_2x_1^2x_4 > 0 \land x_2x_3x_4 < 0 \)

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
\text{apply } \Gamma
\]
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable?

Routh-Hurwitz criterion gives

\[
\begin{align*}
73/62 \ x^2y &> 0 \land 3844x^3(3472x^2 + 7081y^3)y > 0 \land \\
1/62 (3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) &> 0 \\
\land & - 1/87 (-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) > 0
\end{align*}
\]

Combinatorial part: \( x_1^2x_2 > 0 \land x_1^3x_2x_3 > 0 \land x_3^2x_2x_1^2x_4 > 0 \land x_2x_3x_4 < 0 \)

\[
\text{apply } \Gamma \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{reduce} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable? Routh-Hurwitz criterion gives

\[
\frac{73}{62} x^2 y > 0 \land 3844x^3(3472x^2 + 7081y^3)y > 0 \land \\
\frac{1}{62}(3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) > 0 \\
\land -\frac{1}{87}(-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) > 0
\]

Combinatorial part: \( x_1^2x_2 > 0 \land x_1^3x_2x_3 > 0 \land x_3^2x_2x_1^2x_4 > 0 \land x_2x_3x_4 < 0 \)

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\xrightarrow{\Gamma}
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

apply \( \Gamma^{-1} \): \( x_1 < 0 \land x_2 > 0 \land x_3 < 0 \land x_4 > 0 \)
Example application of main theorem

Is \( f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4 \) Hurwitz stable?

Routh-Hurwitz criterion gives

\[
\begin{align*}
73/62 \ x^2y & > 0 \land 3844x^3(3472x^2 + 7081y^3)y > 0 \land \\
1/62 (3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) & > 0 \\
\land -1/87 (-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) & > 0
\end{align*}
\]

Combinatorial part: \( x_1^2 x_2 > 0 \land x_1^3 x_2 x_3 > 0 \land x_3^2 x_2 x_1^2 x_4 > 0 \land x_2 x_3 x_4 < 0 \)

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

apply \( \Gamma^{-1} \): \( x_1 < 0 \land x_2 > 0 \land x_3 < 0 \land x_4 > 0 \)

\[
x < 0 \land y > 0 \land 3472x^2 + 7081y^3 < 0 \land -336784y^3x^2 - 686857y^6 + 334428x^3
\]
Main theorem and NP-Hardness of simplification

**Problem:** \textsc{Strict Monomial Inequality Conjunction Simplification}

**Instance:** $F$, a conjunction of strict monomial inequalities, and $d$, a non-negative integer.

**Question:** Is there a conjunction $F'$ of strict monomial inequalities with sum-of-total-degrees $\leq d$ such that $F' \iff F$ under the assumption that all variables are non-zero?
Main theorem and NP-Hardness of simplification

**Problem:** MINIMUM-DISTANCE
**Instance:** $m n$-dimensional vectors $b_1, \ldots, b_m$ over $GF(2)$ and $d$, a non-negative integer.
**Question:** Is there an element of the vector space spanned by $b_1, \ldots, b_n$ of Hamming weight at most $d$?
(Proven NP-Complete in Vardy 1997)

↓ reduce to

**Problem:** STRICT MONOMIAL INEQUALITY CONJUNCTION SIMPLIFICATION
**Instance:** $F$, a conjunction of strict monomial inequalities, and $d$, a non-negative integer.
**Question:** Is there a conjunction $F'$ of strict monomial inequalities with sum-of-total-degrees $\leq d$ such that $F' \iff F$ under the assumption that all variables are non-zero?
Conclusion and Future Work

Conclusion

- Paper extends algorithms to mixed strict/non-strict inequalities.
- Paper gives algorithm for deducing equations implied by conjunctions of monomial inequalities.

Future Work

- Implement and combine with fast methods for making sign deductions (as in the example problem).
- Integrate with algorithms such as Weisspffenning’s virtual term substitution or CAD to make deductions and simplifications during algorithm execution.
- Dealing with disjunction.