Fast Simplications for Tarski Formulas

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Computing with real polynomial inequalities/equalities

$$S := (2x + y - 1)(2x - y - 1) < 0 \ \wedge \ x^2 + y^2 - 1 \leq 0 \ \vee \ x - 1 = 0$$

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Some kinds of questions we might want to answer

SAT	QE	Optimization
Is S satisfiable?	Describe x s.t. $\exists y[S]$	maximize $\frac{xy-1}{x^2+y^2}$ over S
Yes, for example at $(-13/16, 1/4)$	$x + 1 \ge 0 \wedge x + 4/9 \le 0 \vee 17x^2 + 16x + 3 < 0 \vee x - 1 = 0$	$(\sqrt{2} - 1)/2$

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Problems with symbolic methods:

- slow
- memory intensive

- sensitive to input presentation
- overly complex results

Simplification of polynomial eq's and ineq's

Kinds: SAT is simplification; minimze w.r.t. to some metric; more explicit is simpler; “I know it when I see it”

Prior Work: Limited!

- Hong 1992
- Dolzman & Sturm 1997
- Brown 1999

My goal: Simplify quickly! We need *algorithms* for simplifictions that *can* be done quickly, and *proofs* for simplifications that can’t.

This paper: Defines subclass of simplifications, provides efficient algorithms for some, and intractability proofs for others.

Using propositional logic to simplify inequalities

$$x^2 - 3xyz > 0 \wedge z^3y + x < 0 \wedge y^2z^2 - 2x + 1 > 0$$
$$\vee$$
$$x^2 - 3xyz \leq 0 \wedge z^3y + x < 0 \wedge y^2z^2 - 2x + 1 > 0$$

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$A \wedge B \wedge C \ \vee \ \neg A \wedge B \wedge C$

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$A \wedge B \wedge C \ \vee \ \neg A \wedge B \wedge C$

$\downarrow$ simplify

$B \wedge C$

$\downarrow$ map back

$z^3 y + x < 0 \wedge y^2 z^2 - 2x + 1 > 0$

Using propositional logic to simplify inequalities (cont.)

The good

- Satisfiability and simplification for propositional logic is well-studied; algorithms exist with good practical performance
- The inequalities are complete black boxes — time to check SAT or simplify is independent of degree, bit-size, and other metrics for size of the individual inequalities. No algebraic computations!

The bad

- Deductions require repetitions of inequalities (or their negations)
- The inequalities are complete black boxes — cannot make deductions based on them.
- SAT is NP-Complete, Simplification is Σ_2^P-Complete.

Big idea: map to the theory of "monomial inequalities"

$$(x+y)^3 x < 0 \quad \wedge \quad (x+y)^2 x(z^2 - 2xy) < 0 \quad \wedge \quad (x+y)x^5(z^2 - 2xy) > 0$$

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Restriction to conjunctions of monomial inequalities

- Allowing disjunctions means our SAT problem is NP-Hard, and simplification is Σ_2^P-Hard.
- Expressiveness of conjunctions of monomial inequalities lies strictly between that of conjunctions of single-variable sign conditions and of arbitrary boolean combinations of sign-conditions.
    - $x > 0 \wedge y > 0 \vee x = 0$ cannot be expressed as a conjunction of monomial inequalities
    - $xy < 0$ cannot be expressed as a conjunction of single-variable sign conditions.
- Conjunction of monomial-inequalities is often a compact representation.
- Simplifications still useful for formulas with disjunction.

Results for conjunctions of monomial inequalities

- Algorithm for SAT for conjunctions of monomial inequalities; runs in (low degree) polynomial time.
- Proof that simplification of conjunctions of monomial inequalities (minimizing sum of total-degrees) is NP-Hard (decision-problem variant is NP-Complete)
- Proof that simplifying a single monomial inequality $p \sigma 0$ assuming some conjunction of monomial inequalities F holds is NP-Hard.
- Algorithm for determining all single-variable sign-conditions implied by a given conjunction of monomial inequalities; runs in (low degree) polynomial time.
- Algorithm for determining all equalities implied by a given conjunction of monomial inequalities; runs in (low degree) polynomial time.

Restricting to strict monomial inequalities

Consider the following mapping from strict monomial inequalities in $\{x_1, \ldots, x_n\}$ and $GF(2)^{n+1}$:

$$\Gamma\left(\prod_{j=1}^{n} x_j^{d_j} \ \sigma\ 0\right) = [B_1, \ldots, B_{n+1}],$$

where $B_j = d_j \bmod 2$, for $1 \leq j \leq n$,
and $B_{n+1} = 0$ if σ is $>$ and 1 if σ is $<$.

Examples:

- $\Gamma(x_1 x_3 < 0) = [1, 0, 1, 1]$
- $\Gamma(x_1 x_3 > 0) = [1, 0, 1, 0]$
- $\Gamma(x_1^2 x_2^3 x_3 < 0) = [0, 1, 1, 1]$
- $\Gamma^{-1}([1, 0, 1, 1]) = x_1 x_3 < 0$
- $\Gamma^{-1}([0, 0, 0, 0]) = 1 > 0$
- $\Gamma^{-1}([0, 0, 0, 1]) = 1 < 0$

Main theorem

Theorem: Let $F = A_1 \wedge \cdots \wedge A_m$, where each A_i is a monomial inequality over $\{x_1, \ldots, x_n\}$, and let $b_i = \Gamma(A_i)$. Assuming that F is satisfiable over $\mathbb{R}$,

1. $[0, \ldots, 0, 1] \notin \text{span}(b_1, \ldots, b_m)$, and
2. $b \in \text{span}(b_1, \ldots, b_m)$ if and only if $\forall x_1, \ldots, x_n[F \Rightarrow \Gamma^{-1}(b)]$.

Corollary: F is satisfiable over $\mathbb{R}$ iff $[0, \ldots, 0, 1] \notin \text{span}(b_1, \ldots, b_m)$.

Example application of main theorem

Is $f = 87xy - 56xyz^2 - 62x^2z^3 + 97xy^3z - 73yz^4$ Hurwitz stable?

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Routh-Hurwitz criterion gives

$$73/62\ x^2y > 0\ \wedge\ 3844x^3(3472x^2 + 7081y^3)y > 0\ \wedge$$
$$1/62\ (3472x^2 + 7081y^3)^2yx^2(-336784y^3x^2 - 686857y^6 + 334428x^3) > 0$$
$$\wedge\ -1/87\ (-336784y^3x^2 - 686857y^6 + 334428x^3)((3472x^2 + 7081y^3)y) > 0$$

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Combinatorial part: $x_1^2x_2 > 0 \wedge x_1^3x_2x_3 > 0 \wedge x_3^2x_2x_1^2x_4 > 0 \wedge x_2x_3x_4 < 0$

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$$\xrightarrow{\text{apply } \Gamma} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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$x < 0 \wedge y > 0 \wedge 3472x^2 + 7081y^3 < 0 \wedge -336784y^3x^2 - 686857y^6 + 334428x$

Main theorem and NP-Hardness of simplification

Problem: STRICT MONOMIAL INEQUALITY CONJUNCTION SIMPLIFICATION

Instance: F, a conjunction of strict monomial inequalities, and d, a non-negative integer.

Question: Is there a conjunction F' of strict monomial inequalities with sum-of-total-degrees $\leq d$ such that $F' \Leftrightarrow F$ under the assumption that all variables are non-zero?

Main theorem and NP-Hardness of simplification

Problem: MINIMUM-DISTANCE

Instance: m n-dimensional vectors $b_1, \ldots, b_m$ over $GF(2)$ and d, a non-negative integer.

Question: Is there an element of the vector space spanned by $b_1, \ldots, b_n$ of Hamming weight at most d?
(Proven NP-Complete in Vardy 1997)

$\downarrow$ reduce to

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Conclusion and Future Work

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- Paper extends algorithms to mixed strict/non-strict inequalities.
- Paper gives algorithm for deducing equations implied by conjunctions of monomial inequalities.

Future Work

- Implement and combine with fast methods for making sign deductions (as in the example problem).
- Integrate with algorithms such as Weisspfenning's virtual term substitution or CAD to make deductions and simplifications during algorithm execution.
- Dealing with disjunction.