Fast Algorithms for Differential Equations in Positive Characteristic

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ISSAC 2009, KIAS, Seoul, Korea, July 30, 2009

Main objects & Aim

- $\mathbb{F}_p$ = the finite field with p elements (p = prime number)
- $L = \ell_0(x) + \ell_1(x)\partial + \cdots + \ell_r(x)\partial^r$ of order r in $\mathbb{F}_p[x]\langle\partial\rangle$

Def: *p*-curvature $\mathbf{A}_p$ of L = the matrix in $\mathscr{M}_r(\mathbb{F}_p(x))$ whose j-th column contains the coefficients of ∂^{p+j-1} mod L for $1 \leq j \leq r$

Aim of this paper: design efficient algorithms for computing

- the *polynomial solutions* of L
- the *p-curvature* $\mathbf{A}_p$ of L

- Efficiency = complexity estimates with a low exponent in p

Basics on differential equations in characteristic p

- Main differences between characteristic zero and p
  1. (Honda 1981) solutions are simpler in characteristic p

$$\mathcal{S}_L(\mathbb{F}_p[x]) = \mathcal{S}_L(\mathbb{F}_p(x)) = \mathcal{S}_L(\mathbb{F}_p[[x]]) =: \mathcal{S}_L$$

  2. Cauchy's theorem does not hold: the dimension of $\mathcal{S}_L$ over the field of constants $C = \mathbb{F}_p(x^p)$ is generally $< r = \mathrm{ord}(L)$

Example: $y' = y$ has no solution in $\mathbb{F}_p[[x]]$

- Connection between solutions and p-curvature

Theorem. (Katz & Cartier 1970) $\mathrm{rank}(\mathbf{A}_p) = r - \dim_C(\mathcal{S}_L)$

$\longrightarrow$ p-curvature measures to what extent $\dim_C(\mathcal{S}_L)$ is close to r

Motivation

- van der Put 1995: p-curvature publicised in computer algebra, as a tool for factoring operators in $\mathbb{F}_p(x)\langle\partial\rangle$
- Cluzeau 2003: first complexity analysis and implementation of van der Put's algorithms; extension to systems
- Cluzeau, van Hoeij 2004: polynomial solutions mod p and p-curvature used as filters in modular algorithms for $\mathbb{Q}(x)\langle\partial\rangle$

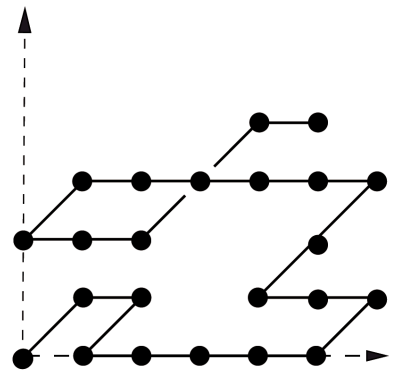
- Concrete applications:
  - enumerative combinatorics (classification of lattice walks)
  - statistical physics (square lattice Ising model)

Combinatorial application: Gessel's conjecture

- Gessel walks: walks in $\mathbb{N}^2$ using only steps in $\mathcal{S} = \{\nearrow, \swarrow, \leftarrow, \rightarrow\}$
- $g(i, j, n)$ = number of walks from $(0, 0)$ to (i, j) with n steps in $\mathcal{S}$

Question: Nature of the generating function

$$G(x, y, t) = \sum_{i,j,n=0}^{\infty} g(i, j, n)\, x^i y^j t^n \in \mathbb{Q}[[x, y, t]]$$



Theorem. (B. & Kauers 2008) $G(x, y, t)$ is an algebraic function.[†]

$\longrightarrow$ Effective, computer-driven discovery and proof

$\longrightarrow$ Key step in discovery: p-curvature computation of two 11th order (guessed) differential operators for $G(x, 0, t)$, and $G(0, y, t)$

[†] The MinPoly $P(x, y, t, G(x, y, t)) = 0$ has $> 10^{11}$ monomials; $\approx$30Gb (!)

Previous work

① p-curvature: size $\mathcal{O}(p)$, but complexity $\mathcal{O}(p^2)$

▶ Main difficulty: non-commutativity of $\mathbb{F}_p(x)\langle\partial\rangle$ prevents from using binary powering techniques for $\mathbf{A}_p$ via $\partial^p \bmod L$

• Katz 1982: first algorithm, based on the matrix recurrence

$$\mathbf{A}_1 = \mathbf{A}, \quad \mathbf{A}_{k+1} = \mathbf{A}'_k + \mathbf{A} \cdot \mathbf{A}_k,$$

where $\mathbf{A} \in \mathscr{M}_r(\mathbb{F}_p(x))$ is the companion matrix associated to L

• van der Put, Cluzeau: variants, all of complexity $\mathcal{O}(p^2)$

② Polynomial solutions mod p: few algorithms in the literature

• Cluzeau 2003: general algorithm of complexity $\mathcal{O}(p^3)$; different $\mathcal{O}(p^2)$ algorithm in the special case $\mathbf{A}_p = 0$

New results

1. on the space $\mathcal{S}_L$ of polynomial solutions

(1.a) degree bound linear in p for all elements in a basis of $\mathcal{S}_L$

(1.b) testing if $\mathcal{S}_L = 0$ in time $\tilde{\mathcal{O}}(\sqrt{p})$

(1.c) computing a whole basis of $\mathcal{S}_L$ in time $\tilde{\mathcal{O}}(p)$

2. on computing the p-curvature $\mathbf{A}_p$

(2.a) for first order operators in time $\mathcal{O}(\log(p))$

(2.b) for certain second order operators in time $\tilde{\mathcal{O}}(p)$

(2.c) for any operator in time $\tilde{\mathcal{O}}(p^{1.8})$

(2.d) deciding nilpotency of $\mathbf{A}_p$ for second order operators in $\tilde{\mathcal{O}}(\sqrt{p})$

p-curvature of arbitrary operators

Theorem. The p-curvature of any L in $\mathbb{F}_p[x]\langle\partial\rangle$ can be computed in subquadratic time $\tilde{\mathcal{O}}(p^{1+\frac{\omega}{3}}) \subset \mathcal{O}(p^{1.8})$.

If $\mathbf{A} = \text{CompanionMatrix}(L)$ and $\Lambda = \partial + \mathbf{A}$, then $\mathbf{A}_p = \Lambda^{p-1}(\mathbf{A})$.

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① Compute $\Gamma = \Lambda^k$ by binary powering.

Basic operation: product in bidegree (k, k) in $\mathbb{F}_p(x)\langle\partial\rangle$.

Cost: $\mathcal{O}(k^\omega)$ (B., Chyzak & Le Roux'08)

② Compute $\mathbf{A}_{(1)} = \mathbf{A}$, $\mathbf{A}_{(i)} = \Gamma\mathbf{A}_{(i-1)}$, $i = 2, \ldots, \ell = (p-1)/k$.

Basic operation: $L(f)$ with $\text{bideg}_{(x,\partial)}(L) = (k, k)$ and $\deg(f) \leq ik$.

Cost: $\tilde{\mathcal{O}}(\ell^{\omega-1}k^2)$ (see next slide)

③ Return $\mathbf{A}_p = \mathbf{A}_{(\ell)}$.

Total cost: $\tilde{\mathcal{O}}(p^{1+\frac{\omega}{3}})$ obtained for $k \approx p^{2/3}$.

Fast evaluation of differential operators

Theorem. Given $L \in \mathbb{F}_p[x]\langle\partial\rangle$ of bidegree (k,k) and $f \in \mathbb{F}_p[x]$ of degree ik, $(i \le s := \sqrt{k})$, one can compute Lf in time $\tilde{\mathcal{O}}(i^{\omega-2}k^2)$.

Algo [baby steps / giant steps strategy inspired by Brent-Kung'78]

① (baby steps) Compute $f_0 = f,\ f_1 = \partial f,\ \ldots,\ f_{s-1} = \partial^{s-1} f$

② (rewriting) Cut L into s slices of bidegree (k,s) in (x,∂):

$$L = L_0 + \partial^s L_1 + \cdots + \partial^{(s-1)s} L_{s-1}$$

③ (recombination) Deduce $L_0 f, \ldots, L_{s-1} f$ at once by a product of polynomial matrices of sizes $(s,s) \times (s,i)$ and degree k

④ (giant steps) Compute and return $Lf = \sum_{0 \le j < s} \partial^{js} L_j f$

Cost: $\tilde{\mathcal{O}}(ik^{3/2})$ for ① and ④; $\tilde{\mathcal{O}}(k^2)$ for ② and $\tilde{\mathcal{O}}(i^{\omega-2}k^2)$ for ③

Computing polynomial solutions (I)

- $\mathcal{S}_L =$ the C-vector space of polynomial solutions of $Lf = 0$
- $\mathcal{G} =$ the $\mathbb{F}_p$-vector space $\mathcal{S}_L \cap \mathbb{F}_p[x]_{<pd}$ where $d = \max(\deg(\ell_i))$

Theorem. $\mathcal{S}_L$ admits a C-basis included in $\mathcal{G}$.

Algorithm for computing $\mathcal{S}_L$:

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① Decide if $\mathcal{S}_L = 0$ ($\Leftrightarrow$ decide if $\mathcal{G} = 0$). If so, stop.
② If not, compute a $\mathbb{F}_p$-basis $(f_1, \ldots, f_k)$ of $\mathcal{G}$.
③ $(f_1, \ldots, f_k)$ generates $\mathcal{S}_L$ over C. Extract a basis.

Cost: $\tilde{\mathcal{O}}(\sqrt{p})$ for ① and $\tilde{\mathcal{O}}(p)$ for ② and ③

Computing polynomial solutions (II)

Pb: Compute an $\mathbb{F}_p$-basis of sols $f = \sum_{i=0}^{pd-1} c_i x^i \in \mathbb{F}_p[x]$ of $Lf = 0$.

- Band-diagonal linear system (S1), width $\mathcal{O}(1) \rightarrow$ gradual solving
- Technical difficulty: some rightmost band elements can be zero!

Algorithm [generalization of (ABP1995) & (BCluzeauSalvy2005)]

① From (S1), deduce an equivalent system (S2) of size $\mathcal{O}(1)$

Basic operation: *matrix factorial* $C(p-1)\cdots C(r)$

Cost: $\tilde{\mathcal{O}}(\sqrt{p})$ (Chudnovsky^2 1989)

② From a basis of (S2), deduce a basis of (S1)

Basic operation: forward substitution

Cost: $\mathcal{O}(p)$

Example

$$L = (5x^2+4)\partial^2 + (4x^2+6x+5)\partial + 2x + 2 \in \mathbb{F}_7[x]\langle\partial\rangle$$

Polynomial solution $f = \sum_{i=0}^{7\cdot 2-1} c_i x^i$ in $\mathbb{F}_7[x]$ s.t. $Lf = 0$:

$$\text{(S1)} \left\{ \begin{array}{c} 2c_0+5c_1+c_2=0, \\ 2c_0+c_1+3c_2+3c_3=0, \\ 6c_1+3c_2+c_3+6c_4=0, \\ 3c_2+c_3+6c_4+3c_5=0, \\ 2c_4+4c_5+c_6=0, \\ 4c_4+6c_5+2c_6=0, \\ c_5+6c_6=0, \\ 5c_6+2c_7+5c_8+c_9=0, \\ 2c_7+c_8+3c_9+3c_{10}=0, \\ 6c_8+3c_9+c_{10}+6c_{11}=0, \\ 3c_9+c_{10}+6c_{11}+3c_{12}=0, \\ 2c_{11}+4c_{12}+c_{13}=0, \\ 4c_{11}+6c_{12}+2c_{13}=0, \\ c_{12}+6c_{13}=0, \\ 5c_{13}=0. \end{array} \right. \Leftrightarrow \begin{bmatrix} 2 & 5 & 1 & & & & & & & & & & & \\ 2 & 1 & 3 & 3 & & & & & & & & & & \\ 0 & 6 & 3 & 1 & 6 & & & & & & & & & \\ & 0 & 3 & 1 & 6 & 3 & & & & & & & & \\ & & 0 & 0 & 2 & 4 & 1 & & & & & & & \\ & & & 0 & 4 & 6 & 2 & 0 & & & & & & \\ & & & & 0 & 1 & 6 & 0 & 0 & & & & & \\ & & & & & 0 & 5 & 2 & 5 & 1 & & & & \\ & & & & & & 0 & 2 & 1 & 3 & 3 & & & \\ & & & & & & & 0 & 6 & 3 & 1 & 6 & & \\ & & & & & & & & 0 & 3 & 1 & 6 & 3 & \\ & & & & & & & & & 0 & 0 & 2 & 4 & 1 \\ & & & & & & & & & & 0 & 4 & 6 & 2 \\ & & & & & & & & & & & 0 & 1 & 6 \\ & & & & & & & & & & & & 0 & 5 \end{bmatrix} \times \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = 0$$

➥ Rewriting the equations in red using the unknowns in blue yields

$$c_4 = 5c_1,\ c_5 = 2c_1,\ c_6 = 3c_1,\ c_{11} = 5c_8 + 5c_1,\ c_{12} = 2c_8,\ c_{13} = 3c_8 + 4c_1$$

➥ Plugging them back into the red equations gives

$$\text{(S2)}: \qquad 3c_1 = 0, 6c_1 = 0, 3c_8 = 0, c_8 + 6c_1 = 0, 6c_8 + 3c_1 = 0$$

➥ $\dim(\text{S1}) = \dim(\text{S2}) = 2$ and Basis(S1) is obtained by subst. from Basis(S2) $= \{(c_0, c_1, c_7, c_8) = (1,0,0,0), (c_0, c_1, c_7, c_8) = (0,0,1,0)\}$

p-curvature of 2nd order operators

- p-curvature $\mathbf{A}_p(L) = (a_{i,j})$ satisfies a *matrix differential equation*:

$$\mathbf{A}_p' = \mathbf{A}_p \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{A}_p \qquad \text{(Cluzeau, 2003)}$$

where $\mathbf{A}$ is the companion matrix of L.

- If $L = v\partial^2 + w\partial + u \in \mathbb{F}_p[x]\langle\partial\rangle$, this is equivalent to the system

$$\begin{aligned} v^3 a_{2,1}''' + A a_{2,1}' + B a_{2,1} &= 0, \qquad (\dagger) \\ v^2 a_{1,2} + R a_{2,1}'' + S a_{2,1}' + T a_{2,1} &= 0, \\ v(a_{1,1} - a_{2,2}) + v a_{2,1}' - w a_{2,1} &= 0, \\ a_{1,1}' + a_{2,2}' &= 0, \end{aligned}$$

where $A, B, R, S, T \in \mathbb{F}_p[x]$ are explicit in terms of v, w, u.

- computing $\mathbf{A}_p$ amounts to solving Eq $(\dagger)$, in time $\tilde{\mathcal{O}}(p)$.

Conclusion, open questions

So far:

- algorithm of quasi-optimal complexity for solving $Lf = 0$.

Still open:

- compute the p-curvature in quasi-linear time (at least for second order operators!)
- decide in sublinear time, e.g. in $\tilde{\mathcal{O}}(\sqrt{p})$, if $\mathbf{A}_p(L)$ is nilpotent when $\mathrm{ord}(L) > 2$.
- are the problems of computing p-curvature and computing polynomial solutions computationally equivalent?