Algorithms for Regular Solutions of Higher-Order Linear Differential Systems

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Problem

System of $n$ linear differential equations of order $\ell$:

$$\mathcal{L}(\vartheta)y(x) = A_\ell(x)\vartheta^\ell y(x) + A_{\ell-1}(x)\vartheta^{\ell-1}y(x) + \cdots + A_0(x)y(x) = 0,$$

where

- $\vartheta = x \frac{d}{dx}$,
- for $i = 0, \ldots, \ell$, $A_i(x) = \sum_{j=0}^{\infty} A_{i,j} x^j \in \mathbb{K}[x]^{n \times n}$ ($\mathbb{K} \subseteq \mathbb{C}$), such that $A_\ell(0) = A_{\ell,0}$ is invertible.

Problem: Compute a basis of the formal regular solutions space, i.e., formal solutions of the form $y(x) = x^{\lambda}z(x)$ where

- $\lambda \in \overline{\mathbb{K}}$, ($\overline{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$),
- $z(x) \in \overline{\mathbb{K}}[[x]]^n[\ln(x)]$. 
1. Introduction
   - Existing methods
   - Our approach

2. Systems with constant matrix coefficients
   - Matrix polynomials
   - Homogeneous systems
   - Non-homogeneous systems with particular right-hand side

3. Systems with variable matrix coefficients
   - BCE Algorithm (Generalization of Poole’s method)
   - Experimental results
   - Case where $A_\ell(0)$ is not necessarily invertible

4. Contributions
1 Introduction
   - Existing methods
   - Our approach

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   - Homogeneous systems
   - Non-homogeneous systems with particular right-hand side

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4 Contributions
Existing methods

- **Classical method**: convert to 1st order system of dimension $n\ell$
  \[ \theta Y = CY, \]  
  where $C$ is a bloc companion matrix with entries in $\mathbb{K}[[x]]$.

  → Coddington & Levinson (1955),

  **Drawback**: increase the size of the system, $n \rightarrow n\ell$.

- **Direct methods**:
    Lack of this method: suppose that the eigenvalues of the matrix $C(0)$ do not differ by integers.
  - Abramov et al (2005): generalization of Heffter’s algorithm
    → associated matrix recurrence system
Poole’s method for the scalar case \( n = 1 \) (1936)

\[
\mathcal{L}(\vartheta)y(x) = a_\ell(x)\vartheta^\ell y(x) + a_{\ell-1}(x)\vartheta^{\ell-1}y(x) + \cdots + a_0(x)y(x) = 0.
\]

Look for solution of the form

\[
y(x) = x^{\lambda_0} \left( u_0 + u_1 x + u_2 x^2 + \cdots \right)
\]

where \( \lambda_0 \in \overline{K} \) and \( u_i \in \overline{K}[\ln(x)] \).

Plugging \( y(x) \) in \( \mathcal{L}(\vartheta)y(x) = 0 \), we obtain that:

- \( \lambda_0 \) must be a root of the indicial polynomial

\[
L_0(\lambda) = a_\ell(0)\lambda^\ell + a_{\ell-1}(0)\lambda^{\ell-1} + \cdots + a_0(0) \in K[\lambda]
\]

- the \( u_i \) satisfy

\[
L_0(\vartheta + \lambda_0 + i)u_i = q_i(\ln(x))
\]

where \( q_i(\ln(x)) \in K(\lambda_0)[\ln(x)] \).
We need to

- consider **matrix polynomials** which will play the same role as indicial polynomials
- look for **polynomial solutions** in $\ln(x)$ of systems with constant matrix coefficients of the form

$$L(\vartheta)y(x) = \left(A_\ell \vartheta^\ell + A_{\ell-1} \vartheta^{\ell-1} + \cdots + A_0 \right) y(x) = Q(\ln(x)),$$

where $A_i \in \mathbb{K}^{n \times n}$ and $Q(\ln(x)) \in \mathbb{K}(\alpha)[\ln(x)]^n$. 
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Contributions
Matrix polynomials

\[ L(\lambda) = A_\ell \lambda^\ell + A_{\ell-1} \lambda^{\ell-1} + \cdots + A_0 \]

where \( A_i \in \mathbb{K}^{n \times n} \) and \( A_\ell \) is invertible.

- \( \deg(\det(L(\lambda))) = n \ell. \)
- \( \lambda_0 \in \overline{\mathbb{K}} \) eigenvalue of \( L(\lambda) \) if \( \det(L(\lambda_0)) = 0. \)
  Its multiplicity \( m_a(\lambda_0) \) is called the algebraic multiplicity of \( \lambda_0. \)
- \( m_g(\lambda_0) := \dim(\ker(L(\lambda_0))) \) is called the geometric multiplicity of \( \lambda_0. \)
- \( v_0 \neq 0 \in \overline{\mathbb{K}}^n \) eigenvector associated to \( \lambda_0 \) if \( L(\lambda_0)v_0 = 0. \)
Let $L(\lambda) \in \mathbb{K}[\lambda]^{n \times n}$ and $\lambda_0 \in \bar{\mathbb{K}}$ an eigenvalue.
There exist $E_{\lambda_0}(\lambda), F_{\lambda_0}(\lambda) \in \bar{\mathbb{K}}[\lambda]^{n \times n}$ invertible at $\lambda_0$, such that

$$L(\lambda) = E_{\lambda_0}(\lambda) \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & (\lambda - \lambda_0)^{\kappa_1} & \\ & & & \ddots \\ & & & & (\lambda - \lambda_0)^{\kappa_{mg(\lambda_0)}} \end{pmatrix} F_{\lambda_0}(\lambda)$$

- $\kappa_i \in \mathbb{N}^*$ satisfying $0 < \kappa_1 \leq \ldots \leq \kappa_{mg(\lambda_0)}$, unique and called the partial multiplicities associated to $\lambda_0$
- $m_a(\lambda_0) = \sum_{i=1}^{mg(\lambda_0)} \kappa_i$. 

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Jordan chains

Let $L(\lambda) \in \mathbb{K}[\lambda]^{n \times n}$ and $\lambda_0 \in \overline{\mathbb{K}}$ an eigenvalue. A sequence of vectors of $\overline{\mathbb{K}}^n$ $0 \neq v_0, v_1, \ldots, v_{k-1}$ satisfying

$$
\begin{pmatrix}
L(\lambda_0) & 0 & \ldots & 0 \\
\frac{L'(\lambda_0)}{1!} & L(\lambda_0) & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
\frac{L^{(k-1)}(\lambda_0)}{(k-1)!} & \ldots & \ldots & L(\lambda_0)
\end{pmatrix}
\begin{pmatrix}
v_0 \\
v_1 \\
\vdots \\
v_{k-1}
\end{pmatrix} = 0
$$

is called a **Jordan chain of length** $k$ associated to $\lambda_0$.

- The maximum lengths of Jordan Chains associated to $\lambda_0$ are equal to the partial multiplicities $\kappa_i$ of $\lambda_0$.

- **Algorithm for computing partial multiplicities and Jordan chains:** Zúñiga (2005)
  
  **Arithmetic complexity:** $O(n^5 \ell^2 d_{\lambda_0})$ operations in $\mathbb{K}$.
Algorithm 1  Euler matrix differential equation

Input:  \( A_0, \ldots, A_\ell \in \mathbb{K}^{n \times n} \)

Output: A basis of the regular solution space of \( L(\vartheta)y = 0 \)

1. Compute \( \sigma(L) := \{ \lambda_0 \in \overline{\mathbb{K}} \text{ s.t. } \lambda_0 \text{ is an eigenvalue of } L(\lambda) \} \).

2. For each \( \lambda_0 \in \sigma(L) \)
   1. Compute \( \kappa_1, \ldots, \kappa_{m_g(\lambda_0)} \) and the Jordan chains \( v_{i,0}, \ldots, v_{i,\kappa_i-1} \) for \( i = 1, \ldots, m_g(\lambda_0) \) associated to \( \lambda_0 \).
   2. For \( i = 1, \ldots, m_g(\lambda_0) \) and \( j = 0, \ldots, \kappa_i - 1 \), let \( y_{\lambda_0,i,j}(x) = x^{\lambda_0} \sum_{k=0}^{j} v_{i,j-k} \frac{\ln^k(x)}{k!} \).

3. Return all the \( y_{\lambda_0,i,j} \) computed in Step 2.2.

Arithmetic complexity: \( \mathcal{O}(n^5 \ell^2 d_{\lambda_0}) \) operations in \( \mathbb{K} \) for each \( \lambda_0 \in \sigma(L) \).
Theorem

Consider a non-homogeneous system

\[ L(\vartheta)y = A_\ell \vartheta^\ell y + A_{\ell-1} \vartheta^{\ell-1} y + \cdots + A_0 y = Q(\ln(x)), \]

where for \( i = 0, \ldots, \ell \), \( A_i \in \mathbb{K}^{n \times n} \), \( A_\ell \) is invertible and \( Q(\ln(x)) \in \mathbb{K}(\alpha)[\ln(x)]^n \) of degree \( d \). Then there exists at least a polynomial solution in \( \ln(x) \) of degree \( p \) with

\[
d \leq p \leq d + \max\{\kappa_i, i = 1, \ldots, m_g(0)\} \quad \text{if} \quad 0 \in \sigma(L(\lambda)),
\]

\[
p = d \quad \text{if} \quad 0 \notin \sigma(L(\lambda)).
\]
Computing a polynomial solution

If $y(x) = \sum_{i=0}^{p} y_i \frac{\ln^i(x)}{i!}$, $Q(\ln(x)) = \sum_{i=0}^{d} q_i \frac{\ln^i(x)}{i!}$ where $y_i, q_i \in \mathbb{K}(\alpha)^n$
then the $y_i$ verify

$$
\begin{pmatrix}
A_0 \\
\vdots \\
A_{p-d} \\
\vdots \\
A_p \\
\end{pmatrix}
\begin{pmatrix}
y_p \\
y_d \\
y_0 \\
\end{pmatrix}
=
\begin{pmatrix}
0 \\
\vdots \\
0 \\
q_d \\
q_0 \\
\end{pmatrix},
$$

where $A_i = 0$ if $i > \ell$. 
Algorithm 2 Non-homogeneous systems

Input: \( A_0, \ldots, A_\ell \in \mathbb{K}^{n \times n} \) and \( Q(\ln(x)) = \sum_{i=0}^{d} q_i \frac{\ln^i(x)}{i!} \in \mathbb{K}(\alpha)[\ln(x)]^n \).

Output: A polynomial solution in \( \ln(x) \) of \( L(\vartheta)y = Q(\ln(x)) \).

1. If \( 0 \notin \sigma(L) \), then \( L(0) = A_0 \) is invertible and solve the system recursively.

2. Otherwise, compute an \( LU \) decomposition of \( A_0 \) and \( \kappa = \max\{\kappa_i, i = 1, \ldots, m_g(0)\} \). Let \( m := d + \kappa \), \( P := \emptyset \) and \( S := 0 \).

   For \( i = m, \ldots, 0 \),
   \begin{enumerate}
   \item Compute a solution of \( A_0y_i = S \) using \( LU \).
   \item Update \( P = P \cup \{\text{parameters of } y_i\} \) and \( y_{i+1}, \ldots, y_m \).
   \item \( S := q_{i-1} - \sum_{j=1}^{m-i+1} A_jy_{i+j-1} \) where \( q_{i-1} = 0 \) if \( i - 1 > d \).
   \end{enumerate}

3. Return \( y(x) = \sum_{i=0}^{m} y_i \frac{\ln^i(x)}{i!} \).

Arithmetic complexity: \( \mathcal{O}(n^5 \ell^2 d_\alpha) \) operations in \( \mathbb{K} \).
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Contributions
Generalization of Poole’s method

\[ \mathcal{L}(\vartheta)y(x) = A_\ell(x)\vartheta^\ell y(x) + A_{\ell-1}(x)\vartheta^{\ell-1}y(x) + \cdots + A_0(x)y(x) = 0. \]

For \( j \geq 0 \), let \( L_j(\lambda) = A_{\ell,j}\lambda^\ell + \cdots + A_{1,j}\lambda + A_{0,j} \), so that:

\[ \mathcal{L}(\vartheta)y(x) = \sum_{j=0}^{\infty} x^j L_j(\vartheta)y(x) = 0. \]

Want to compute a solution of the form

\[ y(x) = x^{\lambda_0} \sum_{i \geq 0} U_i x^i, \]

where \( \lambda_0 \in \overline{K}, \ U_i \in \overline{K}[\ln(x)]^n \) of degree less than \( n\ell \) s.t. \( U_0 \neq 0 \).
Generalization of Poole’s method

Plugging $y(x)$ in $\mathcal{L}(\vartheta)y(x) = 0$, we find

- $L_0(\vartheta)(x^{\lambda_0}U_0) = 0$,

  **Algorithm 1** $\Rightarrow \lambda_0$ eigenvalue of $L_0(\lambda)$, and $U_0 = \sum_{i=0}^{k-1} v_{k-1-i} \frac{\ln^i(x)}{i!}$, where $v_0, \ldots, v_{k-1}$ is a Jordan chain associated to $\lambda_0$,

- and for $m \geq 1$,

  $L_0(\vartheta + \lambda_0 + m)U_m = -\sum_{i=0}^{m-1} L_{m-i}(\vartheta + \lambda_0 + i)U_i$. 

  \[ Q_m(x) \]

If we suppose that $U_0, \ldots, U_{m-1}$ have been computed as vectors of polynomials in $\ln(x)$, then $Q_m(x) \in \mathbb{K}(\lambda_0)[\ln(x)]^n$ $\Rightarrow U_m$ exists and can be computed using **Algorithm 2**.
Algorithm 3 BCE

Input: $A_0(x), \ldots, A_\ell(x) \in \mathbb{K}[[x]]^{n \times n}$ and $\nu \in \mathbb{N}^*$. 
Output: A basis of the regular solution space of $L(\vartheta)y(x) = 0$, where the power series are truncated up to the order $\nu$.

1. Compute $\sigma(L_0)$.
2. For each $\lambda_0 \in \sigma(L_0)$
   1. Compute $\kappa_1, \ldots, \kappa_{m_\vartheta}(\lambda_0)$ and the Jordan chains $v_{i,0}, \ldots, v_{i,\kappa_i-1}$ for $i = 1, \ldots, m_\vartheta(\lambda_0)$ associated to $\lambda_0$.
   2. For $i = 1, \ldots, m_\vartheta(\lambda_0)$ and $j = 0, \ldots, \kappa_i - 1$,
      1. Let $U_{(i,j),0} = \sum_{k=0}^{j} v_{i,j-k} \frac{\ln^k(x)}{k!}$.
      2. For $m = 1, \ldots, \nu$, compute $U_{(i,j),m}$ using Algorithm 2.
3. Return all the $y_{\lambda_0,i,j}(x) = x^{\lambda_0} \sum_{m=0}^{\nu} U_{(i,j),m} x^m$.

Arithmetic complexity: $\mathcal{O}(n^5 \ell^3 \nu^2 d_{\lambda_0})$ operations in $\mathbb{K}$ for each $\lambda_0 \in \sigma(L_0)$. 

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Experimental results: Timings (in seconds) for $n = 2$

- **BCE**: the implementation of our algorithm
- **LFS**: LinearFunctionalSystems (Abramov et al)
- **FOS**: convert to first-order system and use ISOLDE (Barkatou-Pflügel)

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\nu$</th>
<th>$\sigma(L_0)$</th>
<th>BCE</th>
<th>LFS</th>
<th>FOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>${\frac{1}{2}, \frac{5}{2}, \frac{4}{3}}$</td>
<td>0.862</td>
<td>0.677</td>
<td>0.858</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>${\frac{1}{2}, \frac{5}{2}, \frac{4}{3}}$</td>
<td>2.346</td>
<td>6.229</td>
<td>4.671</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>${\frac{1}{2}, \frac{5}{2}, \frac{4}{3}}$</td>
<td>5.619</td>
<td>28.807</td>
<td>21.714</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>${\frac{1}{2}, 1}$</td>
<td>0.498</td>
<td>0.626</td>
<td>1.151</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>${\frac{1}{2}, 1}$</td>
<td>1.490</td>
<td>5.301</td>
<td>5.135</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>${\frac{1}{2}, 1}$</td>
<td>3.625</td>
<td>24.195</td>
<td>19.315</td>
</tr>
</tbody>
</table>

The computations were made on 2.4 GHz Intel Core 2 Duo.
The experimental results show the timings (in seconds) for \( \ell = 4 \) using different values of \( n \) and \( \nu \), with \( \sigma(L_0) = \{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\} \) and \( \{\frac{1}{2}, 1\} \) for different cases.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \nu )</th>
<th>( \sigma(L_0) )</th>
<th>BCE</th>
<th>LFS</th>
<th>FOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}}</td>
<td>0.926</td>
<td>0.865</td>
<td>1.083</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}}</td>
<td>8.263</td>
<td>17.359</td>
<td>24.336</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}}</td>
<td>25.075</td>
<td>95.329</td>
<td>308.903</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>{\frac{1}{2}, 1}</td>
<td>0.462</td>
<td>0.644</td>
<td>1.130</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>{\frac{1}{2}, 1}</td>
<td>2.121</td>
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</tr>
<tr>
<td>8</td>
<td>5</td>
<td>{\frac{1}{2}, 1}</td>
<td>7.282</td>
<td>60.397</td>
<td>246.695</td>
</tr>
</tbody>
</table>

The computations were made on a 2.4 GHz Intel Core 2 Duo.
Case where $A_\ell(0)$ is not necessarily invertible

**Theorem**

Let

$$\mathcal{L}(v)y(x) = A_\ell(x)v^\ell y(x) + A_{\ell-1}(x)v^{\ell-1}y(x) + \cdots + A_0(x)y(x) = 0$$

with

- $A_\ell(x)$ invertible in $\mathbb{K}((x))^{n \times n}$ (and not necessarily at the origin),
- $L_0(\lambda) = A_\ell(0)\lambda^\ell + A_{\ell-1}(0)\lambda^{\ell-1} + \cdots + A_0(0)$.

**If det($L_0(\lambda)$) \neq 0, then the dimension of the regular solution space is exactly equal to deg(det($L_0(\lambda)$)).**

**Consequence:**

- **Algorithm BCE** computes a basis of the regular solution space even if $A_\ell(0)$ is not invertible.
- The system has a regular singularity at the origin $\iff A_\ell(0)$ is regular.
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4 Contributions
Contributions

- Development of an algorithm which solve the problem.
- Complexity analysis and implementation in Maple.
- Extension to the case where $A_\ell(x)$ is invertible (not necessarily at the origin) and $\det(L_0(\lambda)) \neq 0$.
- Generalization of the well-known Frobenius method.
Thanks for your attention !!
Timings (in seconds) for $\ell = 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\nu$</th>
<th>$\sigma(L_0)$</th>
<th>BCE</th>
<th>LFS</th>
<th>FOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>${1, 2}$</td>
<td>0.785</td>
<td>0.177</td>
<td>0.236</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>${1, 2, 3}$</td>
<td>7.098</td>
<td>1.096</td>
<td>1.045</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>${\alpha_0, 0, 1}$</td>
<td>0.821</td>
<td>0.994</td>
<td>0.480</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>${\alpha_0, 0, \ldots, 5}$</td>
<td>7.706</td>
<td>6.730</td>
<td>2.810</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>${1, \frac{1}{2}, \ldots, \frac{1}{4}}$</td>
<td>0.266</td>
<td>0.429</td>
<td>0.584</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>${1, \frac{1}{2}, \ldots, \frac{1}{8}}$</td>
<td>0.920</td>
<td>3.630</td>
<td>3.669</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>${1}$</td>
<td>0.264</td>
<td>0.139</td>
<td>0.255</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>${1}$</td>
<td>0.878</td>
<td>0.692</td>
<td>0.743</td>
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<tr>
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<td>${\alpha_1, \alpha_2}$</td>
<td>1.875</td>
<td>1.666</td>
<td>0.555</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>${\alpha_1, \alpha_2, \alpha_3}$</td>
<td>7.079</td>
<td>5.223</td>
<td>1.711</td>
</tr>
</tbody>
</table>

$\alpha_0 = \text{RootOf}(x^2 + 1)$, $\alpha_1 = \text{RootOf}(x^2 - 2x + 2)$, $\alpha_2 = \text{RootOf}(x^2 - 4x + 5)$ and $\alpha_3 = \text{RootOf}(x^2 - 6x + 10)$. 