Algorithms for Regular Solutions of Higher-Order Linear Differential Systems

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Problem

System of n linear differential equations of order ℓ:

$$\mathcal{L}(\vartheta)y(x) = A_\ell(x)\vartheta^\ell y(x) + A_{\ell-1}(x)\vartheta^{\ell-1}y(x) + \cdots + A_0(x)y(x) = 0,$$

where

- $\vartheta = x\dfrac{d}{dx}$,
- for $i = 0, \ldots, \ell$, $A_i(x) = \sum_{j=0}^{\infty} A_{i,j}x^j \in \mathbb{K}[[x]]^{n\times n}$ ($\mathbb{K} \subseteq \mathbb{C}$), such that $A_\ell(0) = A_{\ell,0}$ is invertible.

Problem: Compute a basis of the formal regular solutions space, *i.e.*, formal solutions of the form $y(x) = x^\lambda z(x)$ where

- $\lambda \in \bar{\mathbb{K}}$, ($\bar{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$),
- $z(x) \in \bar{\mathbb{K}}[[x]]^n[\ln(x)]$.

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Existing methods

- **Classical method**: convert to 1st order system of dimension $n\ell$
  $\vartheta \mathcal{Y} = \mathcal{C}\mathcal{Y}$, where $\mathcal{C}$ is a bloc companion matrix with entries in $\mathbb{K}[[x]]$.

  $\longrightarrow$ Coddington & Levinson (1955),
  $\longrightarrow$ Barkatou & Pflügel (1997, implemented in the package ISOLDE).

  Drawback: increase the size of the system, $n \longrightarrow n\ell$.

- **Direct methods**:
  - Jódar et al (1993-94):
    Lack of this method: suppose that the eigenvalues of the matrix $\mathcal{C}(0)$ do not differ by integers.
  - Abramov et al (2005): generalization of Heffter's algorithm
    $\longrightarrow$ associated matrix recurrence system

Poole's method for the scalar case $n = 1$ (1936)

$$\mathcal{L}(\vartheta)\, y(x) = a_\ell(x)\vartheta^\ell y(x) + a_{\ell-1}(x)\vartheta^{\ell-1} y(x) + \cdots + a_0(x) y(x) = 0.$$

Look for solution of the form

$$y(x) = x^{\lambda_0}\left(u_0 + u_1 x + u_2 x^2 + \cdots\right)$$

where $\lambda_0 \in \bar{\mathbb{K}}$ and $u_i \in \bar{\mathbb{K}}[\ln(x)]$.

Plugging $y(x)$ in $\mathcal{L}(\vartheta)\, y(x) = 0$, we obtain that:

- λ_0 must be a root of the indicial polynomial

$$L_0(\lambda) = a_\ell(0)\lambda^\ell + a_{\ell-1}(0)\lambda^{\ell-1} + \cdots + a_0(0) \in \mathbb{K}[\lambda]$$

- the u_i satisfy

$$L_0(\vartheta + \lambda_0 + i) u_i = q_i(\ln(x))$$

  where $q_i(\ln(x)) \in \mathbb{K}(\lambda_0)[\ln(x)]$.

Generalization to the matrix case

We need to

- consider matrix polynomials which will play the same role as indicial polynomials
- look for polynomial solutions in $\ln(x)$ of systems with constant matrix coefficients of the form

$$L(\vartheta)y(x) = \left(A_\ell \vartheta^\ell + A_{\ell-1}\vartheta^{\ell-1} + \cdots + A_0\right) y(x) = Q(\ln(x)),$$

where $A_i \in \mathbb{K}^{n\times n}$ and $Q(\ln(x)) \in \mathbb{K}(\alpha)[\ln(x)]^n$.

Matrix polynomials

$$L(\lambda) = A_\ell \lambda^\ell + A_{\ell-1}\lambda^{\ell-1} + \cdots + A_0$$

where $A_i \in \mathbb{K}^{n\times n}$ and A_ℓ is invertible.

- $\deg(\det(L(\lambda))) = n\ell$.
- $\lambda_0 \in \bar{\mathbb{K}}$ eigenvalue of $L(\lambda)$ if $\det(L(\lambda_0)) = 0$.
  Its multiplicity $m_a(\lambda_0)$ is called the algebraic multiplicity of λ_0.
- $m_g(\lambda_0) := \dim(\ker(L(\lambda_0)))$ is called the geometric multiplicity of λ_0.
- $v_0 \neq 0 \in \bar{\mathbb{K}}^n$ eigenvector associated to λ_0 if $L(\lambda_0)v_0 = 0$.

Local Smith form - Partial multiplicities

Let $L(\lambda) \in \mathbb{K}[\lambda]^{n\times n}$ and $\lambda_0 \in \bar{\mathbb{K}}$ an eigenvalue.
There exist $E_{\lambda_0}(\lambda), F_{\lambda_0}(\lambda) \in \bar{\mathbb{K}}[\lambda]^{n\times n}$ invertible at λ_0, such that

$$L(\lambda) = E_{\lambda_0}(\lambda) \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & (\lambda-\lambda_0)^{\kappa_1} & & \\ & & & & \ddots & \\ & & & & & (\lambda-\lambda_0)^{\kappa_{m_g(\lambda_0)}} \end{pmatrix} F_{\lambda_0}(\lambda)$$

- $\kappa_i \in \mathbb{N}^*$ satisfying $0 < \kappa_1 \leq \ldots \leq \kappa_{m_g(\lambda_0)}$, unique and called the partial multiplicities associated to λ_0
- $m_a(\lambda_0) = \sum_{i=1}^{m_g(\lambda_0)} \kappa_i$.

Jordan chains

Let $L(\lambda) \in \mathbb{K}[\lambda]^{n \times n}$ and $\lambda_0 \in \bar{\mathbb{K}}$ an eigenvalue. A sequence of vectors of $\bar{\mathbb{K}}^n$ $0 \neq v_0, v_1, \ldots, v_{k-1}$ satisfying

$$\begin{pmatrix} L(\lambda_0) & 0 & \ldots & 0 \\ \frac{L^{'}(\lambda_0)}{1!} & L(\lambda_0) & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \frac{L^{(k-1)}(\lambda_0)}{(k-1)!} & \ldots & \ldots & L(\lambda_0) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \end{pmatrix} = 0$$

is called a Jordan chain of length k associated to λ_0.

- The maximum lengths of Jordan Chains associated to λ_0 are equal to the partial multiplicities κ_i of λ_0.
- Algorithm for computing partial multiplicities and Jordan chains: Zúñiga (2005)
  Arithmetic complexity: $\mathcal{O}(n^5 \ell^2 d_{\lambda_0})$ operations in $\mathbb{K}$.

Algorithm 1 Euler matrix differential equation

Input: $A_0, \ldots, A_\ell \in \mathbb{K}^{n\times n}$
Output: A basis of the regular solution space of $L(\vartheta)y = 0$

1. Compute $\sigma(L) := \{\lambda_0 \in \bar{\mathbb{K}} \text{ s.t. } \lambda_0 \text{ is an eigenvalue of } L(\lambda)\}$.
2. For each $\lambda_0 \in \sigma(L)$
   1. Compute $\kappa_1, \ldots, \kappa_{m_g(\lambda_0)}$ and the Jordan chains $v_{i,0}, \ldots, v_{i,\kappa_i-1}$ for $i = 1, \ldots, m_g(\lambda_0)$ associated to λ_0.
   2. For $i = 1, \ldots, m_g(\lambda_0)$ and $j = 0, \ldots, \kappa_i - 1$, let
   $y_{\lambda_0,i,j}(x) = x^{\lambda_0} \sum_{k=0}^{j} v_{i,j-k} \frac{\ln^k(x)}{k!}$.
3. Return all the $y_{\lambda_0,i,j}$ computed in Step 2.2.

Arithmetic complexity: $\mathcal{O}(n^5\ell^2 d_{\lambda_0})$ operations in $\mathbb{K}$ for each $\lambda_0 \in \sigma(L)$.

Non-homogeneous systems with particular right-hand side

Theorem

Consider a non-homogeneous system

$$L(\vartheta)y = A_\ell \vartheta^\ell y + A_{\ell-1}\vartheta^{\ell-1} y + \cdots + A_0 y = Q(\ln(x)),$$

where for $i = 0, \ldots, \ell$, $A_i \in \mathbb{K}^{n\times n}$, A_ℓ *is invertible and* $Q(\ln(x)) \in \mathbb{K}(\alpha)[\ln(x)]^n$ *of degree* d*. Then there exists at least a polynomial solution in* $\ln(x)$ *of degree* p *with*

$$\begin{array}{ll} d \leq p \leq d + \max\{\kappa_i, i = 1, \ldots, m_g(0)\} & \textit{if} \quad 0 \in \sigma(L(\lambda)), \\ p = d & \textit{if} \quad 0 \notin \sigma(L(\lambda)). \end{array}$$

Computing a polynomial solution

If $y(x) = \sum_{i=0}^{p} y_i \frac{\ln^i(x)}{i!}$, $Q(\ln(x)) = \sum_{i=0}^{d} q_i \frac{\ln^i(x)}{i!}$ where $y_i, q_i \in \mathbb{K}(\alpha)^n$ then the y_i verify

$$\begin{pmatrix} A_0 & & & & \\ \vdots & \ddots & & (0) & \\ A_{p-d} & & \ddots & & \\ \vdots & & & \ddots & \\ A_p & \dots & \dots & \dots & A_0 \end{pmatrix} \begin{pmatrix} y_p \\ \vdots \\ y_d \\ \vdots \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_d \\ \vdots \\ q_0 \end{pmatrix},$$

where $A_i = 0$ if $i > \ell$.

Algorithm 2 Non-homogeneous systems

Input: $A_0, \ldots, A_\ell \in \mathbb{K}^{n\times n}$ and $Q(\ln(x)) = \sum_{i=0}^{d} q_i \frac{\ln^i(x)}{i!} \in \mathbb{K}(\alpha)[\ln(x)]^n$.
Output: A polynomial solution in $\ln(x)$ of $L(\vartheta)y = Q(\ln(x))$.

1. If $0 \notin \sigma(L)$, then $L(0) = A_0$ is invertible and solve the system recursively.
2. Otherwise, compute an *LU* decomposition of A_0 and $\kappa = \max\{\kappa_i, i = 1, \ldots, m_g(0)\}$. Let $m := d + \kappa$, $P := \emptyset$ and $S := 0$. For $i = m, \ldots, 0$,
    1. Compute a solution of $A_0 y_i = S$ using *LU*.
    2. Update $P = P \cup \{\text{parameters of } y_i\}$ and $y_{i+1}, \ldots, y_m$.
    3. $S := q_{i-1} - \sum_{j=1}^{m-i+1} A_j y_{i+j-1}$ where $q_{i-1} = 0$ if $i - 1 > d$.
3. Return $y(x) = \sum_{i=0}^{m} y_i \frac{\ln^i(x)}{i!}$.

Arithmetic complexity: $\mathcal{O}(n^5 \ell^2 d_\alpha)$ operations in $\mathbb{K}$.

Generalization of Poole's method

$$\mathcal{L}(\vartheta)y(x) = A_\ell(x)\vartheta^\ell y(x) + A_{\ell-1}(x)\vartheta^{\ell-1}y(x) + \cdots + A_0(x)y(x) = 0.$$

For $j \geq 0$, let $L_j(\lambda) = A_{\ell,j}\lambda^\ell + \cdots + A_{1,j}\lambda + A_{0,j}$, so that :

$$\mathcal{L}(\vartheta)y(x) = \sum_{j=0}^{\infty} x^j L_j(\vartheta)y(x) = 0.$$

Want to compute a solution of the form

$$y(x) = x^{\lambda_0} \sum_{i \geq 0} U_i x^i,$$

where $\lambda_0 \in \bar{\mathbb{K}}$, $U_i \in \bar{\mathbb{K}}[\ln(x)]^n$ of degree less than $n\ell$ s.t. $U_0 \neq 0$.

Generalization of Poole's method

Plugging $y(x)$ in $\mathcal{L}(\vartheta)y(x) = 0$, we find

- $L_0(\vartheta)\left(x^{\lambda_0} U_0\right) = 0$,

  Algorithm 1 $\Rightarrow$ λ_0 eigenvalue of $L_0(\lambda)$, and $U_0 = \sum_{i=0}^{k-1} v_{k-1-i} \dfrac{\ln^i(x)}{i!}$, where $v_0, \ldots, v_{k-1}$ is a Jordan chain associated to λ_0,
- and for $m \geq 1$,

$$L_0(\vartheta + \lambda_0 + m)U_m = \underbrace{-\sum_{i=0}^{m-1} L_{m-i}(\vartheta + \lambda_0 + i)U_i}_{Q_m(x)}.$$

If we suppose that $U_0, \ldots, U_{m-1}$ have been computed as vectors of polynomials in $\ln(x)$, then $Q_m(x) \in \mathbb{K}(\lambda_0)[\ln(x)]^n$
$\Rightarrow$ U_m exists and can be computed using Algorithm 2.

Algorithm 3 BCE

Input: $A_0(x), \ldots, A_\ell(x) \in \mathbb{K}[[x]]^{n \times n}$ and $\nu \in \mathbb{N}^*$.
Output: A basis of the regular solution space of $\mathcal{L}(\vartheta)y(x) = 0$, where the power series are truncated up to the order ν.

1. Compute $\sigma(L_0)$.
2. For each $\lambda_0 \in \sigma(L_0)$
   1. Compute $\kappa_1, \ldots, \kappa_{m_g(\lambda_0)}$ and the Jordan chains $v_{i,0}, \ldots, v_{i,\kappa_i - 1}$ for $i = 1, \ldots, m_g(\lambda_0)$ associated to λ_0.
   2. For $i = 1, \ldots, m_g(\lambda_0)$ and $j = 0, \ldots, \kappa_i - 1$,
      1. Let $U_{(i,j),0} = \sum_{k=0}^{j} v_{i,j-k} \frac{\ln^k(x)}{k!}$.
      2. For $m = 1, \ldots, \nu$, compute $U_{(i,j),m}$ using Algorithm 2.
3. Return all the $y_{\lambda_0,i,j}(x) = x^{\lambda_0} \sum_{m=0}^{\nu} U_{(i,j),m} x^m$.

Arithmetic complexity: $\mathcal{O}(n^5 \ell^3 \nu^2 d_{\lambda_0})$ operations in $\mathbb{K}$ for each $\lambda_0 \in \sigma(L_0)$.

Experimental results: Timings (in seconds) for $n = 2$

- BCE: the implementation of our algorithm
- LFS: LinearFunctionalSystems (Abramov et al)
- FOS: convert to first-order system and use ISOLDE (Barkatou-Pflügel)

ℓ	ν	$\sigma(L_0)$	BCE	LFS	FOS
4	5	$\{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\}$	0.862	0.677	0.858
8	5	$\{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\}$	2.346	6.229	4.671
12	5	$\{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\}$	5.619	28.807	21.714
4	5	$\{\frac{1}{2}, 1\}$	0.498	0.626	1.151
8	5	$\{\frac{1}{2}, 1\}$	1.490	5.301	5.135
12	5	$\{\frac{1}{2}, 1\}$	3.625	24.195	19.315

The computations were made on 2.4 GHz Intel Core 2 Duo.

Experimental results: Timings (in seconds) for $\ell = 4$

n	ν	$\sigma(L_0)$	BCE	LFS	FOS
2	5	$\{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\}$	0.926	0.865	1.083
5	5	$\{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\}$	8.263	17.359	24.336
8	5	$\{\frac{1}{2}, \frac{5}{2}, \frac{4}{3}\}$	25.075	95.329	308.903
2	5	$\{\frac{1}{2}, 1\}$	0.462	0.644	1.130
5	5	$\{\frac{1}{2}, 1\}$	2.121	10.774	20.228
8	5	$\{\frac{1}{2}, 1\}$	7.282	60.397	246.695

The computations were made on 2.4 GHz Intel Core 2 Duo.

Case where $A_\ell(0)$ is not necessarily invertible

Theorem

Let

$$\mathcal{L}(\vartheta)y(x) = A_\ell(x)\vartheta^\ell y(x) + A_{\ell-1}(x)\vartheta^{\ell-1}y(x) + \cdots + A_0(x)y(x) = 0$$

with

- $A_\ell(x)$ *invertible in* $\mathbb{K}((x))^{n\times n}$ *(and not necessarily at the origin),*
- $L_0(\lambda) = A_\ell(0)\lambda^\ell + A_{\ell-1}(0)\lambda^{\ell-1} + \cdots + A_0(0)$.

If $\det(L_0(\lambda)) \not\equiv 0$, *then the dimension of the regular solution space is exactly equal to* $\deg(\det(L_0(\lambda)))$.

Consequence:

- Algorithm BCE computes a basis of the regular solution space even if $A_\ell(0)$ is not invertible.
- The system has a regular singularity at the origin $\Leftrightarrow$ $A_\ell(0)$ is regular.

Contributions

- Development of an algorithm which solve the problem.
- Complexity analysis and implementation in Maple.
- Extension to the case where $A_\ell(x)$ is invertible (not necessarily at the origin) and $\det(L_0(\lambda)) \not\equiv 0$.
- Generalization of the well-known Frobenius method.

Thanks for your attention !!

Timings (in seconds) for $\ell = 1$

n	ν	$\sigma(L_0)$	BCE	LFS	FOS
4	5	$\{1, 2\}$	0.785	0.177	0.236
8	5	$\{1, 2, 3\}$	7.098	1.096	1.045
4	5	$\{\alpha_0, 0, 1\}$	0.821	0.994	0.480
8	5	$\{\alpha_0, 0, \ldots, 5\}$	7.706	6.730	2.810
4	5	$\{1, \frac{1}{2} \ldots, \frac{1}{4}\}$	0.266	0.429	0.584
8	5	$\{1, \frac{1}{2}, \ldots, \frac{1}{8}\}$	0.920	3.630	3.669
4	5	$\{1\}$	0.264	0.139	0.255
8	5	$\{1\}$	0.878	0.692	0.743
4	5	$\{\alpha_1, \alpha_2\}$	1.875	1.666	0.555
6	5	$\{\alpha_1, \alpha_2, \alpha_3\}$	7.079	5.223	1.711

$\alpha_0 = \mathrm{RootOf}(x^2 + 1)$, $\alpha_1 = \mathrm{RootOf}(x^2 - 2x + 2)$, $\alpha_2 = \mathrm{RootOf}(x^2 - 4x + 5)$ and $\alpha_3 = \mathrm{RootOf}(x^2 - 6x + 10)$.