# Chebyshev Expansions for Solutions of Linear Differential Equations

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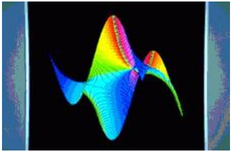
MICROSOFT RESEARCH INRIA JOINT CENTRE



# I Introduction

# How to evaluate a function $f$ in $[-1,1]$?

Two representations of $f$:

- in Taylor series

$$f=\sum_{n=0}^{+\infty} c_n x^n,\ c_n=\frac{f^{(n)}(0)}{n!},$$

- or in Chebyshev series

$$f=\sum_{n=0}^{+\infty} t_n T_n(x),$$

$$t_n=\frac{1}{\pi}\int_{-1}^{1} T_n(t)\frac{f(t)}{\sqrt{1-t^2}}dt.$$

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**Basic properties of Chebyshev polynomials**

$$T_n(cos(\theta)) = \cos(n\theta)$$

$$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = 0 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}$$

$$T_{n+1} = 2xT_n - T_{n-1}$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 4x^3 - 3x$$

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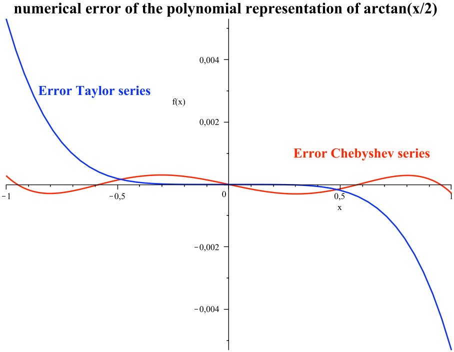


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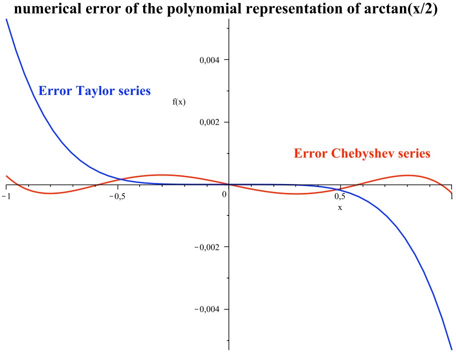


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**How to compute $t_n$?**

General case: numerical computation of the integral. Slow.

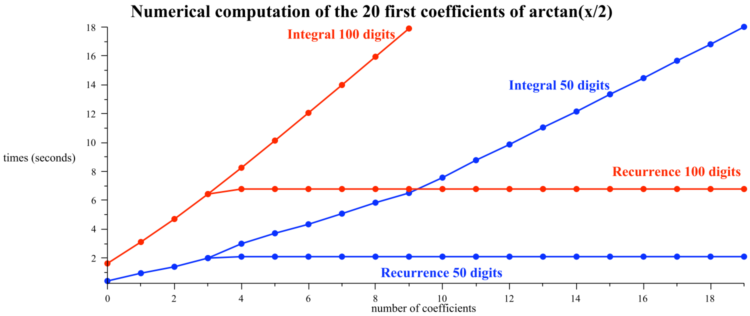
# Computation of Coefficients with Recurrences

**Theorem (60's)**

*If f is solution of a linear differential equation with polynomial coefficients, then the Chebyshev coefficients are cancelled by a linear recurrence with polynomial coefficients.*

Applications:

- Numerical computation of the coefficients.



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Applications:

- Numerical computation of the coefficients.
- Computation of closed-form for coefficients.

**Example ($f(x) = \arctan(x/2)$)**

```
> def:=(4+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x);
```

$$def := (4+x^2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x\left(\frac{d}{dx}y(x)\right) \quad (15)$$

```
> rec:=Chebyshev:-diffeqtorecchebyshev(def,y(x),t(n))=0;
```

$$rec := n\,t(n) + (36+18n)\,t(n+2) + (n+4)\,t(n+4) = 0 \quad (16)$$

```
> r:=simplify(evalc(allvalues(rsolve({rec,seq(t(i)=1/Pi*int(arctan(x/2)*
T(i,x)/sqrt(1-x^2),x=-1..1),i=0..3)},t(n))))) assuming n::integer;
```

$$r := \begin{cases} 0 & n = 0 \\ \dfrac{\left(-2+\sqrt{5}\right)^{n+1}\left(\sqrt{5}+2\right)\sin\left(\frac{1}{2}n\pi\right)}{n} & \textit{otherwise} \end{cases} \quad (17)$$

# State of the Art

- Clenshaw (1957): numerical scheme to compute the Chebyshev coefficients without computing all these integrals.
- Fox and Parker (1968): method for the computation of the Chebyshev recurrence relations for differential equations of small orders.
- Paszkowski (1975): algorithm for computing the Chebyshev recurrence relation.
- Lewanowicz (1976): algorithm for computing a smaller order Chebyshev recurrence relation in some cases.
- Rebillard (1998): new algorithm for computing the Chebyshev recurrence relation.
- Rebillard and Zakrajšek (2006): algorithm for computing a smaller order Chebyshev recurrence relation compared with Lewanowicz algorithm.

# New Results (2009)

- Simple unified presentation of these algorithms using fractions of recurrence operators.
- Complexity analysis of the existing algorithms (order $k$, degree $k$)
  - Paszkowski's and Lewanowicz's algorithms: $O(k^4)$ arithmetic operations in $\mathbb{Q}$.
  - Rebillard's algorithm: $O(k^5)$ arithmetic operations in $\mathbb{Q}$.
- New fast algorithm: $O(k^\omega)$ arithmetic operations. Here, $\omega$ is a feasible exponent for matrix multiplication with coefficients in $\mathbb{Q}$ ($\omega \leq 3$).
- Implementation of algorithm in Maple.

# II Fractions of Recurrence Operators

# Morphisms of Rings of Operators ($S \cdot u_n = u_{n+1}$)

**Taylor series ($f := \sum c_n x^n$)**

$$xf = \sum c_n x^{n+1} = \sum c_{n-1} x^n,$$

$$f' = \sum n c_n x^{n-1} = \sum (n+1) c_{n+1} x^n$$

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$$(4+x^2)\left(\frac{d}{dx}\right)^2 + 2x\frac{d}{dx}$$

$$\mapsto (4+S^{-2})(n+1)(n+2)S^2 + 2S^{-1}(n+1)S$$

$$= (n+1)\left(4(n+2)S^2 + n\right)$$

$$4(n+2)c_{n+2} + nc_n = 0$$

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## Monomial Basis $x^n = M_n(x)$

$$xM_n(x) = M_{n+1}(x),$$
$$(M_n(x))' = nM_{n-1}(x).$$

## Chebyshev series

$$xT_n(x) = 1/2\,(T_{n+1}(x) + T_{n-1}(x))$$
$$T_n'(x) = \frac{n\,(T_{n-1}(x) - T_{n+1}(x))}{2(1-x^2)}.$$

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# Application to Chebyshev recurrences relations

## Definition

Let $\varphi$ be "the Chebyshev morphism":

$$\varphi(x) = \frac{1}{2}\left(S + S^{-1}\right) \text{ et } \varphi\left(\frac{d}{dx}\right) = \frac{2n}{-S + S^{-1}}.$$

## Theorem (BenoitSalvy2009)

*$f \in \mathcal{C}^k$, $L$ is a differential operator of order $k$ such that $L \cdot f = 0$.*
*Suppose that either of the following holds:*

*(i).* $\displaystyle\int_{-1}^{1} \frac{f^{(k)}(x)}{\sqrt{1-x^2}}\,dx$ *is convergent;*

*(ii).* $\displaystyle\int_{-1}^{1} \frac{(1-x^2)^k f^{(k)}(x)}{\sqrt{1-x^2}}\,dx$ *is convergent and* $(1-x^2)^i | p_i$, $i = 0, \dots, k$.

*Then, the Chebyshev coefficients of $f$ are cancelled by a numerator of $\varphi(L)$.*

# Ore Polynomials: Framework for Recurrence Operators

- $\sum a_i(n)u_{n+i}$ is represented by $\sum a_i(n)S^i$.
- These polynomials are non-commutative.
- Multiplication defined by: $Sn = (n+1)S$.
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- Multiplication defined by: $Sn = (n+1)S$.
- Ring denoted $\mathbb{Q}(n)\langle S\rangle$.
- Main property: the degree in $S$ of a product is the sum of the degrees of its factors.
  - Algorithm for (left or right) euclidian division.

**gcld algorithm (Ore 1933)**

INPUT recurrence operators $A$ and $B$
OUTPUT The "greatest" G such that $A = G\tilde{A}$ and $B = G\tilde{B}$

**lclm algorithm (Ore 1933)**

INPUT recurrence operators $A$ and $B$
OUTPUT The "smallest" $U$ and $V$ such that $UA = VB$

# Fractions of Recurrence Operators (Ore 1933)

- Field of fractions of $\mathbb{Q}(n)\langle S\rangle$ defined by:

$$\frac{A}{B} = \frac{C}{D} \Leftrightarrow \exists (U, V) \text{ such that } UA = VC \text{ and } UB = VD.$$

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$$\frac{A}{B} + \frac{C}{D} = \frac{UA}{UB} + \frac{VC}{VD} = \frac{UA + VC}{UB},$$

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- Addition:

$$\frac{A}{B} + \frac{C}{D} = \frac{UA}{UB} + \frac{VC}{VD} = \frac{UA + VC}{UB},$$

- Multiplication:

$$\frac{D}{C} \cdot \frac{A}{B} = \frac{VD}{VC} \cdot \frac{UA}{UB} = \frac{UA}{VC}.$$

# Example: $\sqrt{1-x^2}$

$f=\sqrt{1-x^2}$ is cancelled by the differential operator : $x+(1-x^2)\frac{d}{dx}$.

$$\varphi\left(x+(1-x^2)\frac{d}{dx}\right) = \frac{S+S^{-1}}{2}+\left(1-\frac{S^2+2+S^{-2}}{4}\right)\frac{2n}{-S+S^{-1}}$$

$$=\frac{\left(-S+S^{-1}\right)\left(S+S^{-1}\right)}{2\left(-S+S^{-1}\right)}-\frac{(n+2)S^2+2n-(n-2)S^{-2}}{2\left(-S+S^{-1}\right)}$$

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$$= \frac{-(n+3)S^2+2n-(n-3)S^{-2}}{2\left(-S+S^{-1}\right)}$$

The Chebyshev coefficients $c_n$ satisfy :

$$(n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0.$$

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Example: Normalized fraction for $\sqrt{1-x^2}$

we have:

$$\varphi\left(-x+(-1+x^2)\frac{d}{dx}\right) = \frac{-(n+3)S^2+2n-(n-3)S^{-2}}{2\left(-S+S^{-1}\right)}$$

$$= \frac{\left(-S+S^{-1}\right)\left((n+2)S-(n-2)S^{-1}\right)}{2(-S+S^{-1})}.$$

Smaller order

$$\Rightarrow (n+2)c_{n+1}-(n-2)c_{n-1}=0.$$

# III Algorithms

# Lewanowicz's algorithm (1976)

Horner+Normalize at each step.
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A recurrence verified by the Chebyshev coefficients of $f$ is:

$$(n+2)\,c_{n+1} - (n-2)\,c_{n-1} = 0$$

# Algorithms of Paszkowski (1975) and Rebillard (1998)

Observation: if $D = \varphi(\frac{d}{dx}) = \frac{2n}{-S+S^{-1}}$ then $D^{-1}$ is a polynomial.

- INPUT :

$$L = \sum_{i=0}^{k} p_i(x) \left(\frac{d}{dx}\right)^i$$

- OUTPUT : A numerator of $\varphi(L)$

Computation with polynomials only.

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## Paszkowski

Compute $q_i(x)$ such that

$$\sum_{i=0}^{k} p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^{k} \left(\frac{d}{dx}\right)^i q_i(x).$$

$$\sum_{i=0}^{k} p_i(X) D^i = \frac{\sum_{i=0}^{k} D^{-k+i} q_i(X)}{D^{-k}}.$$

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Compute $q_i(x)$ such that

$$\sum_{i=0}^{k} p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^{k} \left(\frac{d}{dx}\right)^i q_i(x).$$

$$\sum_{i=0}^{k} p_i(X) D^i = \frac{\sum_{i=0}^{k} D^{-k+i} q_i(X)}{D^{-k}}.$$

## Rebillard

$$X_k := D^{-k} X D^k.$$

$$\sum_{i=0}^{k} p_i(X) D^i = \frac{\sum_{i=0}^{k} p_i(X_k) D^{-k+i}}{D^{-k}}.$$

# Our algorithm: Divide and conquer

$D^{-i}$ is of bidegree $(2i, 2i)$.

## New, fast algorithm

Step 1: Compute $q_i(x)$ such that

$$\sum_{i=0}^{k} p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^{k} \left(\frac{d}{dx}\right)^i q_i(x).$$

Step 2 : Divide and conquer

$$\sum_{i=0}^{k} D^{-k+i} q_i(X) =$$

$$D^{-\frac{k}{2}} \sum_{i=0}^{k/2} D^{-\frac{k}{2}+i} q_i(X) + \sum_{i=k/2+1}^{k} D^{-k+i} q_i(X).$$

Balanced products.

# Our algorithm: Divide and conquer

$D^{-i}$ is of bidegree $(2i, 2i)$.

## New, fast algorithm

Step 1: Compute $q_i(x)$ such that

$$\sum_{i=0}^{k} p_i(x)\left(\frac{d}{dx}\right)^i = \sum_{i=0}^{k}\left(\frac{d}{dx}\right)^i q_i(x).$$

Step 2 : Divide and conquer

$$\sum_{i=0}^{k} D^{-k+i} q_i(X) =$$

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Balanced products.

## Theorem

*If the degrees of $p_i$ are at most $k$,*

- *New: $O(k^\omega)$ arithmetic operations.*
- *Paszkowski and Lewanowicz algorithms : $O(k^4)$ arithmetic operations.*
- *Rebillard : $O(k^5)$ arithmetic operations.*

# IV Conclusion and Future Works

# Conclusion and Future works

Contributions:

- Use of fractions of recurrence operators.
- New algorithm.
- Maple code.
- Available in http://ddmf.msr-inria.inria.fr/

Perspectives:

- Numerical computation of the coefficients.
- Compare our algorithm with the algorithm of Rebillard and Zakrajšek.
- Recurrence in other bases (Jacobi, Hermite and Laguerre polynomials, Bessel functions)

```
> diffeqtorecgegenbauer(y(x)-diff(y(x),x),y(x),alpha,u(n));
```

$$(-n-2-\alpha)\,u(n)+\left(2\,n^2+4\,n+4\,\alpha\,n+4\,\alpha+2\,\alpha^2\right)u(n+1)+(n+\alpha)\,u(n+2)$$